18.0 (Fall '13) PSet 4 solutions

Exercise 1. Problem 3 from 3.6. Find a basis for each of the four subspaces associated with A:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution. The reduced row echelon matrix has two pivots. That means the dimensions of $C(A)$ and $C(A^T)$ has to be 2. It follows from here that the dimension of $N(A)$ is $5-2=3$, and the dimension of $N(A^T) = 3-2 = 1$.

The pivot columns are second and fourth ones. The corresponding columns $(1,1,0)^T$ and $(3,4,1)^T$ from A form a basis in the column space $C(A)$. The non-zero rows in the reduced row echelon form, $(0,1,2,3,4)$ and $(0,0,0,1,2)$, form a basis in the row space $C(A^T)$.

The nullspace basis is $(1,0,0,0,0)$, $(0,-2,1,0,0)$, and $(0,2,0,-2,1)$. The left nullspace bases is $(0,-1,1)$.

Exercise 2. Problem 17 from 3.6. Describe the four subspaces of $\mathbb{R}^3$ associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. Both matrices are in the upper triangular form, so we know the pivots. The first matrix has 2 pivots and the second matrix has 3 pivots.

The spaces for the first matrix:

- $C(A) = ax + by$, all vectors with the last coordinate equal to 0.
- $C(A^T) = ay + bz$, all vectors with the first coordinate equal to 0.
- $N(A) = ax$, all vectors with the last two coordinates equal to 0.
- $N(A^T) = az$, all vectors with the first two coordinates equal to 0.

The spaces for the second matrix:

- $C(I + A) = ax + by + cz$, the whole space.
- $C((I + A)^T) = ax + by + cz$, the whole space..
- $N(I + A) = 0$.
- $N((I + A)^T) = 0$.

Exercise 3. Problem 27 from 3.6. If a, b, c are given with $a \neq 0$, how would you choose d so that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has rank 1? Find a basis for the row space and nullspace. Show they are perpendicular!

Solution. To have rank 1, given that the first row is non-zero, the second row should be a multiple of the first row. That is $d = cb/a$. The row space and nullspace should have dimension 1. The first row (a, b) forms the basis of the row space. The nullspace is generated by $(b, -a)$. To show that these two spaces are perpendicular we need to show that the dot product of these two vectors is zero: $(a, b) \cdot (b, -a) = ab - ba = 0$.

Exercise 4. Problem 22 from 3.6. Construct $A = uv^T + wz^T$ whose column space has basis $(1, 2, 4)$, $(2, 2, 1)$ and whose row space has basis $(1, 0)$, $(1, 1)$. Write A as (3 by 2) times (2 by 2).

Solution. We know that uv^T has rank 1 and column space generated by u and row space generated by v. So it might be natural to chose $(1, 2, 4)^T(1, 0) + (2, 2, 1)^T(1, 1)$ as A:

$$A = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 2 \\ 5 & 1 \end{bmatrix}.$$

We can always write A as

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

but it is prettier if we do this:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Exercise 5. Problem 4 from 8.2. (This problem relates to the triangle graph drawn on page 428.) Choose a vector (b_1, b_2, b_3) for which $Ax = b$ can be solved, and another vector b that allows no solution. How are those b's related to $y = (1, -1, 1)$?

Solution. The incidence matrix A for the given graph is:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

We can observe directly that in every column the middle value is the sum of the other two values. Or, more intelligently, we can use the Kirchhoff's Law: go around the loop and write down the equation: $b_1 + b_3 - b_2 = 0$.

Hence, for any vector b that satisfies the above property there is a solution. For example, $b = (1, -1, 0)$. An example of vector b without a solution is $(1, 0, 0)$. Vectors b for which the equation can be solved are perpendicular to $y = (1, -1, 1)$.

Exercise 6. Problem 17 from 8.2 Suppose A is a 12 by 9 incidence matrix from a connected (but unknown) graph.

(a) How many columns of A are independent?

(b) What condition on f makes it possible to solve $A^T y = f$?

(c) The diagonal entries of $A^T A$ give the number of edges into each node. What is the sum of those diagonal entries?

Solution.

(a) The nullspace of an incidence matrix of a connected graph is always one-dimensional. Hence, the rank of this matrix is 8, and there are 8 independent columns.

(b) f must belong to the column space of A^T, which is the same as row space of A. A vector is in the row space if and only if it is perpendicular to $(1, 1, 1, 1, 1, 1, 1, 1, 1)$. Or in other words, if its coefficients sum to zero.

(c) By "the number of edges into each node" the problem means the number of edges coming in and going out. As each edge contributes 2 ends to the sum, the total is $12 \cdot 2 = 24$.

Exercise 7. Problem 11 from 4.1 Draw Figure 4.2 to show each subspace correctly for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

Solution. Both matrices have rank 1. That is the four spaces for each of them has dimension 1, so the drawing should reflect that.

Exercise 8. Problem 28 from 4.1 Why is each of these statements false?

(a) $(1, 1, 1)$ is perpendicular to $(1, 1, -2)$ so the planes $x + y + z = 0$ and $x + y - 2z = 0$ are orthogonal subspaces.

(b) The subspace spanned by $(1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1)$ is the orthogonal complement of the subspace spanned by $(1, -1, 0, 0, 0)$ and $(2, -2, 3, 4, -4)$.

(c) Two subspaces that meet only at the zero vector are orthogonal.

Solution.

(a) Two planes in 3D can not be orthogonal to each other as the sum of their dimensions is greater than 3.

(b) The first subspace is a 2-dimensional subspace in a 5-dimensional space. Its orthogonal complement must be 3-dimensional, so it can not be spanned by two vectors.

(c) For example, consider two lines $x = 0$ and $x + y = 0$ on the plane. These are two lines that intersect at the zero vector, but they are not perpendicular to each other, so they are not orthogonal subspaces.

Exercise 9. Problem 29 from 4.2 If B has rank m (full row rank, independent rows) show that BB^T is invertible.

Solution. We already know that when A has independent columns, $A^T A$ is invertible. Let put $A = B^T$, then A has full column rank, aka has independent columns. Therefore, $A^T A = (B^T)^T B^T = BB^T$ is invertible.

Exercise 10. Problem 11 from 4.2 Project b onto the column space of A by solving $A^T A\hat{x} = A^T b$ and $p = A\hat{x}$:

$$\text{(a) } A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{(b) } A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

Solution.

(a) $p = A(A^T A)^{-1} A^T b = (2, 3, 0)$, $e = (0, 0, 4)$, $A^T e = 0$.

(b) $p = A(A^T A)^{-1} A^T b = (4, 4, 6)$, $e = (0, 0, 0)$, $A^T e = 0$. You might notice that b belongs to the column space of A and answer this question without calculations.