Q1.

Take
$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then
$$E\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z+x \end{pmatrix}.$$

The inverse E^{-1} is

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

with
$$E^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z-x \end{pmatrix}.$$

We have
$$E\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 8 \end{pmatrix}$$

and
$$E^{-1}\begin{pmatrix} 3 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$
again.

Q2.

$$x+4y-2z=1$$
$$x+7y-6z=6$$
$$3y+qz=t.$$

Subtract row 1 from row 2

$$x+4y-2z=1$$
$$3y-4z=5$$
$$3y+qz=t.$$

Subtract row 2 from row 3

$$x+4y-2z=1$$
$$3y-4z=5$$
$$(q+4)z=t-5.$$

The system is singular when $q=-4$. When this happens, the last equation becomes

$$t-5=0$$

So there are no solutions when $t\neq 5$ and infinitely many when $t=5$.

Q2. cont

When $q=-4$, $t=5$ and $z=1$, we must solve

$$3y-4z=5 \quad \therefore \quad y=3$$

$$x+4y-2z=1 \quad \therefore \quad x=-9$$

The solution is $(x,y,z)=(-9,3,1)$.

Q3.

When $x=0$, $y=a$

" $x=1$, $y=a+b+c$

" $x=2$, $y=a+2b$

Thus the equation we must solve is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 12 \end{pmatrix}.$$

The solution is $(a,b,c)=(2,5,-1)$.

Q4.

If

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

then

$$A^2 = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $A^4 = 0$.

Q4 cont.

Then if $v = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$,

$$Av = \begin{pmatrix} 2y \\ 2z \\ 2t \\ 0 \end{pmatrix}, \quad A^2 v = \begin{pmatrix} 4z \\ 4t \\ 0 \\ 0 \end{pmatrix}, \quad A^3 v = \begin{pmatrix} 16t \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and $A^4 v = 0$.

Q5.

a) There are many, but for example

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

b) There are many, but for example

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Q6.

First, apply elimination to $\begin{pmatrix} 10 & 20 \\ 20 & 50 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 10 & 20 \\ 20 & 50 \end{pmatrix} = \begin{pmatrix} 10 & 20 \\ 0 & 10 \end{pmatrix}$$

We must solve

$$\begin{pmatrix} 10 & 20 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\therefore \quad y = -\frac{1}{5}, \quad 10x + 20y = 1$

$\therefore \quad 10x - 4 = 1$

$\therefore \quad x = \frac{1}{2}$

And $\begin{pmatrix} 10 & 20 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} t \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore \quad z = \frac{1}{10}$

$10t + 20z = 0$

$\therefore \quad 10t + 2 = 0 \quad \therefore \quad t = -\frac{1}{5}$

Q6 cont.

So

$$A^{-1} = \begin{pmatrix} x & t \\ y & z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{1}{10} \end{pmatrix}.$$

Q7. Set $E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then

$$E_{21}A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 0 \end{pmatrix}.$$

If $E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$, then

$$E_{32}E_{21}A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{pmatrix} = U$$

an upper triangular matrix.

Q7 cont.

Now
$$E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

and
$$E_{21}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So $A = LU$ where
$$L = E_{21}^{-1} E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

Q8. You will get the matrix

$$\begin{pmatrix} 1 & 2 & 3 & & n-1 & n \\ & 1 & 2 & & & n-1 \\ & & 1 & \ddots & & \\ & & & \ddots & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

with i,j th entry $a_{ij} = \begin{cases} j-i+1 & \text{if } i \leq j \\ 0 & \text{otherwise.} \end{cases}$

Q9. Any correct program + output.

Q10. The BLUE GENE executes

$$\frac{2}{3}\frac{N^3}{T}$$

floating point operations per unit time. Thus the Atari executes

$$\frac{4}{3}\frac{N^3}{T}$$

floating point operations per unit time. To solve a linear system with M inputs, it takes time

$$\left(\frac{2M^3}{3}\right)\Big/\left(\frac{4N^3}{3T}\right) = \frac{T}{2}\frac{M^3}{N^3}.$$

For $\frac{T}{2}\frac{M^3}{N^3} \leq T$, we must have

$$M^3 \leq 2N^3$$

$$\therefore\ M \leq \sqrt[3]{2}\,N.$$