18.06 (Fall '13) Problem Set 10 Solutions

1. (a) Linear. XB is linear since by definition of matrix multiplication each entry of XB is just a linear combination of the entries of X. Similarly, AX is linear. Since a composition of linear transformations is linear, we also have AXB is linear.

   (b) Not linear. Consider any non-zero X. Then $(2X)^T A(2X) = 4X^T AX \neq 2X^T AX$.

   (c) Linear. AX and XB are linear as before, and the sum of linear transformations is linear.

   (d) Linear. The trace is just a linear combination of the entries of X.

   (e) Not linear. Consider $X = I$, the 2 by 2 identity matrix. Then $\det(2I) = 4 \neq 2\det(I)$.

2. Yes, it is linear.

   We have the transformation $T(f(x)) = g(x) = f(x^2 + x)$. This is just saying that our transformation T replaces each x with $x^2 + x$. For linearity, we need to check that $cf(x)$ goes to $cg(x)$ and that $f_1(x) + f_2(x)$ goes to $g_1(x) + g_2(x)$. Clearly,

   $$T(cf(x)) = cf(x^2 + x) = cg(x)$$

   and

   $$T(f_1(x) + f_2(x)) = f_1(x^2 + x) + f_2(x^2 + x) = g_1(x) + g_2(x),$$

   thus this transformation is linear.

3. <u>Problem 37 from 7.2.</u>

   We first find the result of the proposed transformation on each of the input basis "vectors" v_1, v_2, v_3, v_4. These can be written as linear combinations of the same basis "vectors".

   For example,

   $$T(v_1) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = av_1 + cv_3.$$

   Similarly,

   $$T(v_2) = av_2 + cv_4, \quad T(v_3) = bv_1 + dv_3, \quad T(v_4) = bv_2 + dv_4.$$

   Noting that the transformation of the basis "vector" v_i gives us the ith column of A, we conclude

   $$\boxed{A = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}}.$$

4. Problem 35 from 7.2.

The Haar wavelet basis for R^8 is

$$\begin{aligned}
w_1 &= [1,1,1,1,1,1,1,1]^T \\
w_2 &= [1,1,1,1,-1,-1,-1,-1]^T \\
w_3 &= [1,1,-1,-1,0,0,0,0]^T \\
w_4 &= [0,0,0,0,1,1,-1,-1]^T \\
w_5 &= [1,-1,0,0,0,0,0,0]^T \\
w_6 &= [0,0,1,-1,0,0,0,0]^T \\
w_7 &= [0,0,0,0,1,-1,0,0]^T \\
w_8 &= [0,0,0,0,0,0,1,-1]^T.
\end{aligned}$$

Note that these vectors form an orthogonal basis for R^8.

5. Problem 5 from 7.2.

T is a linear transformation from the three-dimensional space V to the three-dimensional space W. $T(v_i)$ is a combination $a_{1i}w_1 + a_{2i}w_2 + a_{3i}w_3$ of the output basis for W. The a's then form the ith column of the matrix A. For example,

$$T(v_1) = 0w_1 + 1w_2 + 0w_3$$

gives the first column: $[0,1,0]^T$. Repeating this we find,

$$T(v_2) = 1w_1 + 0w_2 + 1w_3,$$

$$T(v_3) = 1w_1 + 0w_2 + 1w_3.$$

Therefore,

$$\boxed{A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}},$$

and

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

This is equivalent to the fact that $T(v_1+v_2+v_3) = 2w_1+w_2+2w_3$, which is demonstrable by virtue of the linearity of T:

$T(v_1+v_2+v_3) = T(v_1)+T(v_2)+T(v_3) = (w_2)+(w_1+w_3)+(w_1+w_3) = 2w_1+w_2+2w_3.$

6. Problem 23 from 7.2.

   We require that the matrix $M = \begin{bmatrix} m_1 & m_4 & m_7 \\ m_2 & m_5 & m_8 \\ m_3 & m_6 & m_9 \end{bmatrix}$ is invertible, namely $\det(M) \neq 0$.

   Note: the matrix M represents a change of basis matrix that takes parabolas in the proposed basis v_1, v_2, v_3 to a different (obviously complete) basis for parabolas $w_1 = 1, w_2 = x, w_3 = x^2$. Thus to be able to represent all parabolas from this complete basis (w_1, w_2, w_3) in the proposed basis (v_1, v_2, v_3), we must require that M^{-1} exists.

7. Problem 8 from 9.3.

   To find $|\lambda|_{max}$ we need to find the eigenvalues of the iteration matrix $B = S^{-1}T$.

   For **Jacobi**, $S = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ and $T = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$.

   So we have $B = S^{-1}T = \begin{bmatrix} 1/a & 0 \\ 0 & 1/d \end{bmatrix} \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & b/a \\ c/d & 0 \end{bmatrix}$.

   The characteristic polynomial of B is $\lambda^2 - bc/ad = 0$ which gives $\lambda = \pm(bc/ad)^{1/2}$.

   So $|\lambda| = |(bc/ad)^{1/2}| = |bc/ad|^{1/2}$. Therefore $\boxed{|\lambda|_{max} = |bc/ad|^{1/2}}$.

   For **Gauss-Seidel**, $S = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ and $T = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$.

   So we have $B = S^{-1}T = \begin{bmatrix} 1/a & 0 \\ -c/ad & 1/d \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b/a \\ 0 & -bc/ad \end{bmatrix}$.

   B is upper triangular so we can read the eigenvalues off the diagonal: $\lambda = 0, -bc/ad$.

   So $|\lambda| = 0, |bc/ad|$. Therefore $\boxed{|\lambda|_{max} = |bc/ad|}$.