18.06 F13 Final: solutions

Q1. a) $\begin{pmatrix}2\\3\\0\end{pmatrix} = 3\begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}$ is in the column space of A, and

$$b - \begin{pmatrix}2\\3\\0\end{pmatrix} = \begin{pmatrix}0\\0\\4\end{pmatrix}$$

is orthogonal to the column space of A, so $\begin{pmatrix}2\\3\\0\end{pmatrix}$ is the projection.

b)

$$b = 6\begin{pmatrix}1\\1\\1\end{pmatrix} - 2\begin{pmatrix}1\\1\\0\end{pmatrix}$$ is already in the column space of A, so the projection is just b itself.

Q2 a) By elimination,

A becomes $\begin{pmatrix} z & z & z \\ 0 & c & -c \\ 0 & -c & c \end{pmatrix}$

then $\begin{pmatrix} z & z & z \\ 0 & c & -c \\ 0 & 0 & 0 \end{pmatrix}$

Its pivots are z, c. So A is +ve semidefinite
$\iff z \geq 0, c \geq 0$.

b) We need $3z = 1 \quad \therefore \quad z = \frac{1}{3}$.

We then have all columns summing to 1, but need all entries non-negative

i.e. $z + c = \frac{1}{3} + c \geq 0$, $\frac{1}{3} - c \geq 0$

So $-\frac{1}{3} \leq c \leq \frac{1}{3}$.

Q2. cont

c) A column is free if it is a linear combination of the columns before it. This is only true for the first column if it is $0 \iff z=0$.

d) As we saw in part a), the rank is always ≤ 2. For rank $=2$, we must have $z \neq 0$, $c \neq 0$. (If either is 0, the matrix is rank 1, but if both are $\neq 0$, then

$$(z \ z \ z) \quad \text{and} \quad (0 \ c \ -c) \quad \text{are LI.})$$

e) Suppose $Ax = b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. Then

$$\begin{pmatrix} z & z & z \\ 0 & c & -c \\ 0 & -c & c \end{pmatrix} x' = \begin{pmatrix} b_1 \\ b_2 - b_1 \\ b_3 - b_1 \end{pmatrix} \quad \text{some } x'$$

and
$$\begin{pmatrix} z & z & z \\ 0 & c & -c \\ 0 & 0 & 0 \end{pmatrix} x'' = \begin{pmatrix} b_1 \\ b_2 - b_1 \\ b_2 + b_3 - 2b_1 \end{pmatrix} \quad \text{some } x''$$

So there is a solution if and only if

$$b_2 + b_3 - 2b_1 = 0.$$

Q3 a) They are the solutions to the eqn

$$0 = -\lambda(3-\lambda) - 4 = \lambda^2 - 3\lambda - 4$$
$$= (\lambda+1)(\lambda-4)$$

$\lambda = -1$ or 4.

b) Since A is symmetric, singular values are $|\text{eigenvalues}|$. Smallest SV = min value of $\|Ax\|$ on unit circle $= 1$

Largest SV = max value of $\|Ax\|$ on unit circle

$= 4$

$\therefore \kappa = 4/1 = 4$.

c) Greatest when x is in same direction as v, least when in opposite direction, so

$v = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ max: $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v^T x = 2$

min: $x = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $v^T x = -2$

$v = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ max: $x = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $v^T x = \sqrt{13}$

min: $x = -\frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $v^T x = -\sqrt{13}$

Q3 cont.

d) Max/min $v^T x$ when $v = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ = ~~largest~~
max/min x-coordinate of ellipse

Max/min $v^T x$ when $v = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ =
max/min y-coordinate of ellipse

So: tightest rectangle has corners

$$(\pm\sqrt{13}, \pm 2)$$

Dimensions: $2\sqrt{13} \times 4$.

Q4

a) Since $A^2 = 8I$, any eigenvalue λ of A must satisfy $\lambda^2 = 8$

$\iff \lambda = \pm\sqrt{8}$.

~~We~~ Trace $A = 0$, so eigenvalues sum to 0

$\therefore$ must have 4 eigenvalues of $\sqrt{8}$ and 4 eigenvalues of $-\sqrt{8}$.

Det A = product of eigenvalues

$= (\sqrt{8})^4(-\sqrt{8})^4$

$= 8^4$.

b) All singular values ~~of~~ are $\sqrt{8}$ (since A is symmetric, so singular values = |eigenvalues|)

So $K(A) = \sqrt{8}/\sqrt{8} = 1$.

c) No. No symmetric matrix can have Jordan blocks of size greater than 1.

d) Since det $A \neq 0$, all columns of A are LI

$\Rightarrow$ first 4 columns are LI

$\Rightarrow$ Matrix of 1st 4 columns has rank 4.

Q4 cont

e) Orthogonal complement, since all columns of A are orthogonal to one another.

f) If this number is c, then

$$\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} - \begin{pmatrix}c\\c\\c\\c\\c\\c\\c\\c\end{pmatrix} \quad \text{must be orthogonal to} \quad \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}$$

So $1-8c=0 \quad \Rightarrow \quad c=\frac{1}{8}$.

g) If P is projection onto the first column, then I-P is projection onto last 7 columns. So diagonal entries are $1-\frac{1}{8}=\frac{7}{8}$.

Q5

a)

$$\left\{ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

b) Yes, it's linear: if S is symmetric, then

$(DSD)^T = D^T S^T D^T = DSD$ so DSD symmetric

$D(S+T)D = DSD + DTD$

$D(\lambda S)D = \lambda DSD$

so linear.

$$De_1D = \begin{pmatrix} d^2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$De_2D = \begin{pmatrix} 0 & de \\ de & 0 \end{pmatrix}$$

$$De_3D = \begin{pmatrix} 0 & 0 \\ 0 & e^2 \end{pmatrix}$$

So matrix of T relative to $\{e_1, e_2, e_3\}$ is $\begin{pmatrix} d^2 & 0 & 0 \\ 0 & de & 0 \\ 0 & 0 & e^2 \end{pmatrix}$.

c) No, since ASA need not be symmetric.

e.g. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $T(I) = A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ Not symmetric

Q6 a) Must. M has nullspace of dim 1, and any nonzero nullspace vector x has $Mx=0$.

b) Can't. M has full rank, so there are no nonzero solutions to $Mx=0$.

c) Might. M has dim 1 nullspace, but this may or may not contain vectors with $x_1=1$.

d) Can't. det (permutation matrix) = (sign of permutation) $\neq 0$

(there is exactly one nonzero term in the big formula, and it is ± 1)

e) Might ~~any vector orthogonal to the col. spa~~

e.g. $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ singular

$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ not singular

f) Must. This is the definition of eigenvalue.

$(M-\lambda I)x=0 \iff Mx=\lambda x$.

g) ~~Might. e.g.~~  $\cancel{M=\begin{pmatrix}1 & -1 & 0\\ -1 & 0 & 1\\ 0 & 1 & -1\end{pmatrix}}$

~~not full column rank~~

Can't. Sum of all columns $=0$, so not full column rank.

$\cancel{\bullet \to \bullet}$ $\cancel{M=\begin{pmatrix}1 & -1\end{pmatrix}}$

~~Full column rank~~