18.06 Professor Edelman Final Exam December 19, 2013

Grading

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Your PRINTED name is: ____________________

Please circle your recitation:

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1 (6 pts.)

Project b onto the column space of A:

(a) (3 pts.) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

(b) (3 pts.) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$.

2 (20 pts.)

The matrix $A = \begin{pmatrix} z & z & z \\ z & z+c & z-c \\ z & z-c & z+c \end{pmatrix}$.

a) (4 pts) Under what conditions on z and c would A be positive semidefinite?

b) (4 pts) Under what conditions on z and c would A be Markov?

c) (4 pts) Under what conditions on z and c would the first column of A be a free column?

d) (4 pts) Under what conditions on z and c does A have rank $r = 2$.

e) (4 pts) Assuming A has rank 2, for which b in R^3 does the equation $Ax = b$ have a solution?

3 (22 pts.)

a) (5 pts.) What are the two eigenvalues of $A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$?

b) (5 pts.) What is κ, the ratio of the maximum and minimum values of $\|Ax\|$ over the unit circle $\|x\| = 1$?

c) (6 pts.) What are the maximum and minimum value of $v^T x$, for x over the unit circle $\|x\| = 1$, when $v = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$? Same question when $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$? (Hint: the minimum is negative)

d) (6 pts.) When every point on the unit circle is multiplied by A, the result is an ellipse. Find the dimensions of the tightest rectangle with sides parallel to the coordinate axes that encloses the ellipse. (Hint: the previous maximum and minimum question is meant to be a warm-up. Another hint: singular values are not so useful here.)

4 (23 pts.)

The 8×8 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

is symmetric and satisfies $A^2 = 8I$. The diagonal elements add to 0.

a) (3 pts.) What are the eigenvalues of A and the determinant of A?

b) (2 pts.) What is the condition number $\kappa(A) = \sigma_1(A)/\sigma_8(A)$? (Hint $\kappa \geq 1$ always.)

c) (3 pts.) Can A have any Jordan blocks of size greater than 1? Explain briefly.

d) (2 pts.) What is the rank of the matrix consisting of the first four columns of A?

e) (3 pts.) The subspace of R^8 spanned by the first four columns of A is the ______________ ______________ of the last four columns of A. Fill in the blank with the best two words and explain briefly.

f) (5 pts.) Let P project onto the first column of A. P is an 8×8 matrix all of whose 64 entries are the same number. This number is ____________________________.

g) (5 pts.) Projection onto the last seven columns of A (we are dropping only the first column) gives an 8×8 projection matrix whose 8 diagonal entries are the same number. This number is ____________________________.

5 (15 pts.)

Consider the vector space of symmetric 2×2 matrices of the form

$$S = \begin{pmatrix} x & z \\ z & y \end{pmatrix}.$$

a) (6 pts.) Write down a basis for this vector space.

b) (6 pts.) If D is the diagonal matrix $D = \begin{pmatrix} d & 0 \\ 0 & e \end{pmatrix}$, is $T(S) = DSD$ a linear transformation from this space to itself? If so, write down the matrix of this transformation in your chosen basis.

c) (3 pts.) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we want to know if $T(S) = ASA$ os necessarily a linear transformation from symmetric 2×2 matrices to symmetric 2×2matrices? If yes, explain why. If not, explain why not.

6 (14 pts.)

Choose the best choice from “must", “might", "can’t" (and explain briefly.)

a) (2 pts.) A $3x3$ matrix M with rank $r = 2$ ________________ have a non-zero solution to $Mx = 0$.

b) (2 pts.) A $3x3$ matrix M with rank $r = 3$ ________________ have a non-zero solution to $Mx = 0$ with $x_1 = 0$.

c) (2 pts.) A $3x4$ matrix M with rank $r = 3$ ________________ have a non-zero solution to $Mx = 0$ with $x_1 = 1$.

d) (2 pts.) A permutation matrix ________________ be singular.

e) (2 pts.) A projection matrix ________________ be singular.

f) (2 pts.) $M - \lambda I$ ________________ be singular, if λ is an eigenvalue of M.

g) (2 pts.) We recall the number of columns of an incidence matrix for a graph is the number of nodes and every row has one entry $+1$, one entry -1, and the remaining entries 0. Such an incidence matrix ________________ have full column rank.

Thank you for taking linear algebra. We hope you enjoyed it. Linear algebra will serve you well in the future. Have a happy holiday!