18.06 F13 EXAM 3 SOLUTIONS

Q1. Useful facts:

- Row/column swaps don't alter singular values
- Sum of squares of SVs = sum of squares of entries of matrix
- For symmetric matrix, SVs = |eigenvalues|

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ SVs 0, 0

$\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$ All related to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by row/column swaps. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ symmetric, evals $(1, 0)$ $\Rightarrow$ SVs 1, 0

$\left[\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right]$ All same as $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Rank 1 $\therefore$ one SV is 0. If σ is the other,

$$0^2 + \sigma^2 = 1^2 + 1^2$$

so $\sigma = \sqrt{2}$.

$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]$ Related by col swap. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has evals 1, 1, so SVs 1, 1.

Q1 cont.

$$\left[\begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix}\right]$$

All same as $\begin{pmatrix}1&1\\1&0\end{pmatrix}$, which has evals

$$\frac{1+\sqrt{5}}{2}, \quad \frac{1-\sqrt{5}}{2}.$$

$\frac{1-\sqrt{5}}{2}$ is negative, so SVs are $\frac{\sqrt{5}+1}{2}$, $\frac{\sqrt{5}-1}{2}$.

$\left[\begin{pmatrix}1&1\\1&1\end{pmatrix}\right]$ Symmetric, evals $(0,2)$

$\therefore$ SVs $0, 2$.

Q2. a) No. e.g. $A=I$, $B=-I$, $\frac{A+B}{2}=0$ not invertible.

b) True.

kth column sum of $\frac{A+B}{2}$

$$= \frac{(\text{kth column sum of } A) + (\text{kth column sum of } B)}{2}$$

$$= \frac{1+1}{2} = 1$$

Q2. cont.

c). True.

$$x^T\left(\frac{A+B}{2}\right)x$$

$$= \frac{x^TAx + x^TBx}{2} > 0 \quad \text{if } x^TAx,\ x^TBx > 0$$

d) False. $A=\begin{pmatrix}2&0\\0&0\end{pmatrix}$ and $B=\begin{pmatrix}0&2\\0&2\end{pmatrix}$ are both diagonalizable,

but $\frac{A+B}{2}=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ is not.

e) False. $A=\begin{pmatrix}2&0\\0&0\end{pmatrix}$ and $B=\begin{pmatrix}0&0\\0&2\end{pmatrix}$ are both rank 1,

but $\frac{A+B}{2}=I$ is not.

f) True. In the sum

$$e^A = I + A + \frac{A^2}{2!} + \cdots$$

every term is symmetric, so the sum is symmetric.

g) False. $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is Markov but $e^A=\begin{pmatrix}e&0\\0&e\end{pmatrix}$ is not.

h) True. If $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A, then $e^{\lambda_1}, \cdots, e^{\lambda_n}$ are the eigenvalues of e^A, and e^{λ_i} is positive since λ_i is real.

i) False. $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is singular but $e^A = I$ is not.

j) False. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is orthogonal but $e^A = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$ is not.

Q3. a) $y(t)$ is constant $\Rightarrow \frac{dy}{dt} = 0$.

So $Ay = \frac{dy}{dt} \iff Ay = 0$

$\therefore y \in \text{null } A$.

Can take $y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b) We calculate e^{At}. A has evals $-2, 0$ with corresponding evecs $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So

$A = S \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} S^{-1}$

$At = S \begin{pmatrix} -2t & 0 \\ 0 & 0 \end{pmatrix} S^{-1}$ where $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$e^{At} = S \begin{pmatrix} e^{-2t} & 0 \\ 0 & 1 \end{pmatrix} S^{-1}$

$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & -e^{-2t} \\ 1 & 1 \end{pmatrix}$

Q3. cont

$$= \frac{1}{2}\begin{pmatrix} e^{-2t}+1 & -e^{-2t}+1 \\ -e^{-2t}+1 & e^{-2t}+1 \end{pmatrix}$$

$\frac{1}{2}((e^{-2t}+1)+(-e^{-2t}+1))=1$

$\frac{1}{2}((-e^{-2t}+1)+(e^{-2t}+1))=1$

so Markov.

c) As $t\to\infty$, $e^{-2t}\to 0$, so

$$e^{At} \to \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

d) Steady state of e^A

$$= \lim_{n\to\infty} e^{nA} v$$ where v is any probability vector e.g. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \lim_{t\to\infty} e^{tA} v$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$