Solution Set 9, 18.06 Fall '11

1. Do Problem 5 from 8.3. Surprising?

   Solution. Let $A = \begin{pmatrix} 0.98 & 0 & 0 \\ 0.02 & 0.97 & 0 \\ 0 & 0.03 & 1 \end{pmatrix}$. Since A is a lower triangular matrix, its eigenvalues are its diagonal entries, namely, 0.98, 0.97 and 1. The steady state of this system is the eigenvector $\mathbf{x}$ corresponding to the eigenvalue 1. From

   $$\begin{pmatrix} 0.98-1 & 0 & 0 \\ 0.02 & 0.97-1 & 0 \\ 0 & 0.03 & 1-1 \end{pmatrix} \mathbf{x} = 0,$$

   we get $\mathbf{x} = [0,0,1]^T$, i.e. everyone will be dead eventually. Not quite surprising. □

2. Do Problem 12 from 8.3.

   Solution. The eigenvalues of B are $\lambda_1 = 0$ and $\lambda_2 = -0.5$ if you solve

   $$\det(B - \lambda I) = (-0.2-\lambda)(-0.3-\lambda) - 0.3 \cdot 0.2 = \lambda(\lambda + 0.5) = 0.$$

   Note that A always has eigenvalue 1 so $\det(B) = \det(A - I) = 0$. Thus, B always has eigenvalue 0.

   The corresponding eigenvectors are $\mathbf{x}_1 = [0.3, 0.2]^T$ and $\mathbf{x}_2 = [-1, 1]^T$ and the solution to the given Markov differential equation is

   $$c_1 e^{0 \cdot t} \mathbf{x}_1 + c_2 e^{-0.5t} \mathbf{x}_2 = c_1 \mathbf{x}_1 + c_2 e^{-0.5t} \mathbf{x}_2$$

   As $t \to \infty$, $e^{-0.5t}$ converges to zero so the steady state is $c_1 \mathbf{x}_1$. □

3. Do Problem 3 from 6.3.

   Solution. (a) If every column of A adds to zero, then every row of A^T adds to zero, meaning $A^T [1,1,\ldots,1]^T = 0$. This implies $\det(A^T) = 0$ and, hence, $\det(A) = 0$.

   (b) The eigenvalues of $\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}$ are $\lambda_1 = 0$ and $\lambda_2 = -5$ and the corresponding eigenvectors are $\mathbf{x}_1 = [3,2]^T$ and $\mathbf{x}_2 = [-1,1]^T$. Hence, the general solution of this equation is

   $$\mathbf{u}(t) = C_1 e^{0 \cdot t} [3,2]^T + C_2 e^{-5t} [-1,1]^T = \left[3C_1 - C_2 e^{-5t}, 2C_1 + C_2 e^{-5t}\right]^T.$$

   From $\mathbf{u}(0) = [4,1]^T$, we have

   $$\begin{aligned} 3C_1 - C_2 &= 4 \\ 2C_1 + C_2 &= 1 \end{aligned}$$

and, hence, $C_1 = 1$, $C_2 = -1$. This gives us the solution

$$\mathbf{u}(t) = \left[3 + e^{-5t}, 2 - e^{-5t}\right]^T.$$

The steady state $\mathbf{u}(\infty)$ is $[3,2]^T$ since $e^{-5t} \to 0$ as $t \to \infty$.

□

4. Do Problem 4 from 6.3.

Solution.
$$\frac{d(v+w)}{dt} = \frac{dv}{dt} + \frac{dw}{dt} = (w-v) + (v-w) = 0.$$
This implies that $v + w$ remains constant. Let $\mathbf{u} = [v,w]^T$ then

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{u}.$$

Hence, $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. Solving $\det(A - \lambda I) = (-1-\lambda)^2 - 1 = \lambda(\lambda+2) = 0$, A has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -2$. Corresponding eigenvectors are $\mathbf{x}_1 = [1,1]^T$, $\mathbf{x}_2 = [1,-1]^T$. The geeneral solution has the form

$$\mathbf{u}(t) = C_1\mathbf{x}_1 + C_2 e^{-2t}\mathbf{x}_2.$$

From $\mathbf{u}(0) = [30,10]^T$, we get $C_1 = 20$, $C_2 = 10$ and the solution is

$$\mathbf{u}(t) = \left[20 + 10e^{-2t}, 20 - 10e^{-2t}\right]^T.$$

Hence, $[v(1), w(1)]^T = [20 + 10e^{-2}, 20 - 10e^{-2}]^T$, $[v(\infty), w(\infty)]^T = [20,20]^T$ □

5. Do Problem 5 from 6.3.

Solution. The eigenvalues of $-A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ are 0 and 2. (These are the negatives of the eigenvalues of A.) The solution of the equation in this case is

$$\mathbf{u}(t) = \left[20 + 10e^{2t}, 20 - 10e^{2t}\right]^T,$$

and, hence $v(\infty) = \lim_{t\to\infty} 20 + 10e^{2t}$ diverges to ∞. □

6. Do Problem 8 from 6.4.

Solution. If λ is an eigenvalue of A, then $0 = A^3 x = \lambda^3 \mathbf{x}$ for nonzero eigenvector $\mathbf{x}$ so $\lambda = 0$. Hence, all eigenvalues of A must be zero. For example, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ gives $A^3 = 0$.

On the other hand, if A is symmetric, then A has a diagonalization $A = Q\Lambda Q^T$. Then, $A^3 = Q\Lambda^3 Q^T = 0$ and this implies $\Lambda^3 = 0$ since Q is invertible. Hence, $\Lambda = 0$ and $A = 0$. □

7. Do problem 10 from 6.4.

   Solution. We cannot assume that we have a real eigenvector $\mathbf{x}$. If $\mathbf{x}$ is *not* real, then $\mathbf{x}^T\mathbf{x}$ can vanish for nonzero $\mathbf{x}$ so we cannot divide by $\mathbf{x}^T\mathbf{x}$. For instance, for $\mathbf{x} = [i, 1]^T$, $\mathbf{x}^T\mathbf{x} = 0$. □

8. Do Problem 20 from 6.4 (in some sense, this is the cornerstone of quantum mechanics).

   Solution. $A = \begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}$ is an example of a 2×2 Hermitian matrix.

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1+i \\ 1-i & -1-\lambda \end{pmatrix} = \lambda^2 - 3,$$

   thus the eigenvalues are $\sqrt{3}$, $-\sqrt{3}$, which are real.

   To prove that the eigenvalues of any Hermitian matrix A are real, let λ be an eigenvalue of A and $\mathbf{x}$ be a corresponding eigenvector.

$$\begin{aligned} &A\mathbf{x} = \lambda\mathbf{x} \\ \Rightarrow &\overline{A}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}} \\ \Rightarrow &A^T\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}} \\ \Rightarrow &\overline{\mathbf{x}}^T A = \overline{\lambda}\overline{\mathbf{x}}^T \\ \Rightarrow &\overline{\mathbf{x}}^T A\mathbf{x} = \overline{\lambda}\overline{\mathbf{x}}^T\mathbf{x} \end{aligned}$$

   On the other hand, if we take the inner product with $\mathbf{x}$ on each side of the first equation, we get

$$\overline{\mathbf{x}}^T A\mathbf{x} = \lambda\overline{\mathbf{x}}^T\mathbf{x}.$$

   Hence, $\overline{\lambda}\overline{\mathbf{x}}^T\mathbf{x} = \lambda\overline{\mathbf{x}}^T\mathbf{x}$ and $\overline{\lambda} = \lambda$, i.e. λ is real. (Note that $\overline{\mathbf{x}}^T\mathbf{x}$ is not zero for nonzero $\mathbf{x}$.) □

9. Let $J = \begin{pmatrix} 0.4 & 0.2 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.4 & 0.1 \end{pmatrix}$. This Markov matrix describes surfing behavior in a universe with only four web pages. The $(i.j)$ entry is the probablity that your next browser experience is site i, given that you are currently on j. Note that you can return to the same site again. Using a computer, rank the four web pages in order using the steady state. (you can play with different numbers and consider whether this "pagerank" matches your intuition).

   Solution. [See MATLAB code] □

10. Let A be a fixed 2×2 matrix. Show that all the solutions u to $u' = Au$ form a subspace of the (very big) vector space W of functions $\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$ (you do not need to show that this space of functions is a vector space, but it is good practice to convince yourself). Let V be the subspace of W where each of the $f_i(t)$ above are linear combinations of exponential functions. Give an example of an A where the solutions to $u' = Au$ form a subspace of V (hint: this should be true for most A).

Solution. First of all, $u = 0$ is a solution to $u' = Au$. If u_1, u_2 are two solutions of $u' = Au$, then $(cu_1)' = cu_1' = A(cu_1)$ and $(u_1 + u_2)' = u_1' + u_2' = Au_1 + Au_2 = A(u_1 + u_2)$ so cu_1 and $u_1 + u_2$ are also solutions of $u' = Au$. Hence, all the solutions of $u' = Au$ form a subspace.

For the second part of the problem, take $A = I$, then all the solutions to $u' = u$ have the form

$$\begin{pmatrix} Ce^t \\ De^t \end{pmatrix},$$

so this is a subspace of V. □

MATLAB code

```
%%%%%%%%%%%
%Problem 9%
%%%%%%%%%%%

J=[.4 .2 .2 .3;.3 .5 .3 .5;.1 .2 .1 .1;.1 .1 .4 .1]

J =

    0.4000    0.2000    0.2000    0.3000
    0.3000    0.5000    0.3000    0.5000
    0.1000    0.2000    0.1000    0.1000
    0.1000    0.1000    0.4000    0.1000

sum(J)

ans =

    0.9000    1.0000    1.0000    1.0000


J=[.4 .2 .2 .3;.4 .5 .3 .5;.1 .2 .1 .1;.1 .1 .4 .1]

J =

    0.4000    0.2000    0.2000    0.3000
```

```
    0.4000    0.5000    0.3000    0.5000
    0.1000    0.2000    0.1000    0.1000
    0.1000    0.1000    0.4000    0.1000

sum(J)

ans =

    1.0000    1.0000    1.0000    1.0000

[U,D]=eig(J)

U =

  Columns 1 through 3

   0.4807                -0.8593                 0.2446 + 0.1444i
   0.7972                 0.3021                 0.4376 + 0.1951i
   0.2592                 0.1918                 0.0723 - 0.3395i
   0.2572                 0.3654                -0.7545

  Column 4

   0.2446 - 0.1444i
   0.4376 - 0.1951i
   0.0723 + 0.3395i
  -0.7545


D =

  Columns 1 through 3

   1.0000                      0                      0
        0                 0.1575                      0
        0                      0                -0.0287 + 0.1350i
        0                      0                      0

  Column 4

        0
        0
        0
  -0.0287 - 0.1350i

U(:,1)
```

```
ans =

    0.4807
    0.7972
    0.2592
    0.2572


[u,i]=sort(U(:,1),'descend')

u =

    0.7972
    0.4807
    0.2592
    0.2572


i =

     2
     1
     3
     4

% Ranking is 2,1,3,4
% Note the second row clearly has a lot of weight
% followed by the first row
```