Solution Set 8, 18.06 Fall '11

1. What are the possible eigenvalues of a projection matrix? (Hint: if $P^2 = P$ and v is an eigenvector, look at P^2v and Pv). Show that the values you give are all possible.

   Solution. If $Pv = \lambda v$, $P^2 v = \lambda^2 v = \lambda v$, so $\lambda^2 = \lambda$ and $\lambda = 0$ or 1. 0 and I are two projection matrices with only 0 eigenvalues and 1 eigenvalues, respectively. □

2. What are the possible real eigenvalues of a 4 by 4 permutation matrix? (Hint: consider such a matrix P and powers $I, P, P^2, P^3, \ldots$. Show it eventually has to repeat). You do not need to show the values you give are all possible, but you still must get the correct range to achieve full credit.

   Solution. Consider where P sends $v = (x, y, z, w)^T$. There are only 24 possibilities so the list $v, Pv, \ldots, P^{24}v$ must repeat at some point. If v were an eigenvector with eigenvalue λ, this means $v, \lambda v, \lambda^2 v, \ldots$ must repeat at some point and $\lambda^k = \lambda^{k+l}$for some k and l. Since P is invertible, $\lambda \neq 0$ and we can divide to get $\lambda^l = 1$ for some l. For λ to be real, it must be 1 or -1.

   Alternative solution. Or, note that an $n \times n$ permutation matrix P is orthogonal, $P^T P = I$. For any orthogonal matrix Q, assume $Qv = \lambda v$. Then also $v^T Q^T = \lambda v^T$, and right-multiplying each side by Qv and λv, respectively, and using $Q^T Q = I$, we get $v^T v = v^T Q^T Q v = \lambda^2 v^T v$. But $v^T v \neq 0$ when v is an eigenvector and thus $\lambda^2 = 1$, so $\lambda \in \{\pm 1\}$.

   □

3. (Do this problem for both the permutations $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). Given the matrix M, find the characteristic polynomial $f(x) = det(M - xI)$. In both cases, try to "apply $f(x)$ to M," in the sense that if $f(x) = ax^2 + bx + c$, compute $f(M) = aM^2 + bM + cI$. What happens? (This is called the Cayley-Hamilton Theorem and is always true, though you do not need to prove it. You can use it to prove a lot of cool things.)

   Solution. The identity matrix has the characteristic polynomial $(x-1)^2 = x^2 - 2x + 1$. Applying it to itself gives $I^2 - 2I + I = 0$. The other matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has the characteristic polynomial $x^2 - 1$, and it is easy to check that $M^2 = I$ so $M^2 - I = 0$. Thus, the characteristic polynomial applied to the matrix itself seems to be 0. This is true for any matrix. □

4. Do problem 6 from 6.1.

Solution. The eigenvalues are:

$$\begin{aligned}
&\text{A: Lower triangular (so diagonals) } \lambda_1 = \lambda_2 = 1,\\
&\text{B: Upper triangular, (so diagonals) } \lambda_1 = \lambda_2 = 1,\\
&AB : \det(AB - \lambda I) = \lambda^2 - 4\lambda + 1, \quad \lambda_1 = 2 - \sqrt{3}, \quad \lambda_2 = 2 + \sqrt{3},\\
&BA : \det(BA - \lambda I) = \lambda^2 - 4\lambda + 1, \quad \lambda_1 = 2 - \sqrt{3}, \quad \lambda_2 = 2 + \sqrt{3}.
\end{aligned}$$

(a) No.
(b) Yes. This is true whenever one, say, A is *invertible* (as indeed it was above). Then (and only then) we may write $AB = A(BA)A^{-1}$, meaning that the matrices AB and BA are similar (and similar matrices have the same eigenvalues).

However, if both A and B are singular there can be different geometric multiplicities, as the example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ shows, since $AB = B$ but $BA = 0$. □

5. Do problem 11 from 6.1.

Solution. Since x_1 is an eigenvector for eigenvalue λ_1 of A, it is also an eigenvector for eigenvalue 0 of $A - \lambda_1 I$ because

$$(A - \lambda_1 I)x_1 = Ax_1 - \lambda_1 x_1 = \lambda_1 x_1 - \lambda_1 x_1 = 0.$$

As $x_1 \neq 0$ (eigenvectors are never 0) this shows that $A - \lambda_1 I$ is singular and therefore it's column space is at most one dimensional. Since $x_2 \neq 0$ all that is left to do is to show that x_2 is in $C(A - \lambda_1 I)$, for then the column space will have to be the span of x_2, so that the columns of $A - \lambda_1 I$ will be multiples of x_2. But the following calculation shows that $(\lambda_2 - \lambda_1)x_2$ is in $C(A - \lambda_1 I)$ (remember, the column space of a matrix B consists of all vectors Bx for varying x):

$$(A - \lambda_1 I)x_2 = Ax_2 - \lambda_1 x_2 = (\lambda_2 - \lambda_1)x_2.$$

As $\lambda_2 - \lambda_1 \neq 0$ by assumption, this means that $x_2 \in C(A - \lambda_1 I)$, and this finishes the proof as observed earlier. □

6. Do problem 19 from 6.1.

Solution. (a) Yes. Since the three eigenvalues $0, 1, 2$ are all different, their corresponding 3 eigenvectors are a linearly independent collection, hence they form a basis of $\mathbb{R}^3$. Therefore the null space of B is the span of the eigenvector corresponding to eigenvalue 0. Hence: $\text{rank}(B) = 2$.
(b) Yes. $\det(B) = 0 \cdot 1 \cdot 2 = 0$, so also $\det(B^T B) = \det(B)\det(B^T) = (\det(B))^2 = 0$.
(c) No. Upper triangular counterexamples suffice. Let:

$$B_a = \begin{bmatrix} 0 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad a \in \mathbb{R}$$

Each B_a is upper diagonal, so we read off the eigenvalues: $0, 1, 2$ (regardless of which $a \in \mathbb{R}$). But

$$B_a^T B_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1+a^2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

has the eigenvalues $0, 1+a^2, 4$ (which do depend on the value of a). So, clearly there can be no way in the world to deduce the eigenvalues of $B^T B$ from being told only the eigenvalues of B.

(d) Yes. The eigenvalues of B^2 are $0^2, 1^2, 2^2 = 0, 1, 4$. Since

$$\det\big((B^2+I) - \lambda I\big) = \det(B^2 + (1-\lambda)I),$$

we see that λ is an eigenvalue of $B^2 + I$ exactly when $1 - \lambda$ is an eigenvalue of B^2, that is when $(1-\lambda) \in \{0, 1, 4\}$. Hence $B^2 + I$ has eigenvalues $1, 2, 5$.

Since 0 is thus not an eigenvalue of B^2+I, it is invertible and $(B^2+I)^{-1}$ makes sense to write(!).

Therefore, finally: $(B^2+I)^{-1}$ has eigenvalues $1, 1/2, 1/5$. □

7. Do problem 4 from 6.2.

Solution. (a) False. The zero matrix has linearly independent eigenvectors (any basis of $\mathbb{R}^n$ would do) but is not invertible.

(b) True. Explained on p. 298.

(c) True. A square matrix having linearly independent columns is invertible.

(d) False. For instance S could end up being $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (for instance, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then the columns of S are linearly independent eigenvectors of A) which is not diagonalizable because it doesn't have enough independent eigenvectors: if $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an eigenvector of S then $\begin{bmatrix} x+y \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$ and we must have $\lambda \neq 0$, so that $\lambda = 1$ (consider the cases $y = 0$, $y \neq 0$) and $y = 0$, i.e., the eigenvectors span a line through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

□

8. Do Problem 10 from 6.2 (you must use linear algebra somehow).

Solution. The Fibonnaci sequence begins $F_0 = 0$, $F_1 = 1$ and is determined by the recurrence relation $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$. In other words, if we set $v_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ we have $v_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and

$$v_{n+1} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = Av_n,$$

where we set $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Iterating this relation shows that $v_n = A^n v_0$. Therefore, to get hold of Fibonnaci numbers all we have to do is to get hold of the powers of A^n. To do the latter there is a standard procedure that you have learned in class: diagonalize A! Indeed, if $A = S\Lambda S^{-1}$ with Λ a diagonal matrix then $A^n = S\Lambda^n S^{-1}$ is easy to compute (and hence to show that the second coordinate of v_{3n} is even, which is what we want). However, we can proceed yet more sneakily!

Indeed, the entries of matrices A^n are all integers and since all we care about is parity we may as well treat those entries as residues obtained by division by 2 (we say the entries are residues modulo 2). In fact, while computing A^n doing $n-1$ matrix multipliciations we may as well do the arithmetic operations modulo 2 because all we care about in the end is parity of the entries of A^{3n}. With this caveat in mind, we compute first several matrices in the sequence $I, A, A^2, A^3, \ldots$ (remember, all arithmetic is modulo 2):

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ldots$$

At this point we see that the sequence will start repeating (each time we're multiplying by A so each next element depends only on the previous one). In other words, A^{3n} modulo 2 is I, the 2×2 identity matrix, because we found that the sequence of powers above is 3-periodic. Now we can compute v_{3n} modulo 2:

$$v_{3n} = A^{3n} v_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and as $v_{3n} = \begin{bmatrix} F_{3n+1} \\ F_{3n} \end{bmatrix}$ comparing coordinates we are able to conclude that F_{3n} modulo 2 is zero, i.e., every third Fibonacci number is even! $\square$

9. Do Problems 11 and 12 from 6.2.

Solution.

<u>*Problem 6.2.11*</u>

(a) True. 0 is not an eigenvalue.

(b) False. The below matrix A is a counterexample (inspired by Monsieur Jordan):

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

For this A, the eigenvalue 2 is a double root of the characteristic polynomial, but the geometric multiplicity of the eigenvalue 2 is only one (the 2-eigenspace is spanned by $[1, 0, 0]$).

(c) False. The diagonal matrix with the diagonal elements $2, 2, 5$ is ... diagonal.

Problem 6.2.12

Note first that this exercise must talk about a 2×2 matrix.

(a) False. By using the elementary row operation $E = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$ we easily make a matrix such that $E[1,4] = [1,0]$. Then we can use the usual Jordan example $J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ again (where the only eigenvalue is 2, and the only eigenvectors are the span of $[1,0]$), to make a counterexample via similarity:

$$A_0 = E^{-1} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} E = \begin{bmatrix} -2 & 1 \\ -16 & 6 \end{bmatrix}.$$

It's easy to check that for this A_0 that the only eigenvectors are multiples of $[1,4]$. But $\det A_0 = 2 \cdot 2 \neq 0$, so A_0 is invertible.

(b) True. The characteristic polynomial is 2nd order, hence has two roots. If the roots were different, the two corresponding non-zero eigenvectors would have been linearly independent contradicting that all eigenvectors are in the span of $[1,4]$.

(c) True. There must be a basis of eigenvectors in order to diagonalize, but we have only a one-dimensional eigenspace.

$\square$

10. Do Problem 7 from 8.3 (including the challenge problem!).

Solution. Since all entries are ≥ 0 and each column sums to 1, this A is a Markov matrix. Thus we know that $\lambda_1 = 1$ is an eigenvalue. Since $\mathrm{tr}(A) = \lambda_1 + \lambda_2 = 3/2$, we conclude $\lambda_2 = 1/2$ is another eigenvalue. We diagonalize it using the matrix S of eigenvectors:

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 2/3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & -3/5 \end{bmatrix}.$$

Thus as $k \to +\infty$,

$$A^k = \begin{bmatrix} 1 & 1 \\ 2/3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1/2)^k \end{bmatrix} \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & -3/5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 2/3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & -3/5 \end{bmatrix} = A^\infty.$$

This last matrix product equals $\begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$.

Challenge problem Consider a general 2×2 Markov matrix (note we made sure columns sum to one):

$$M_{a,b} = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix},$$

for arbitrary $0 \leq a \leq 1$ and $0 \leq b \leq 1$.

We are asked to determine all pairs (a, b) for which the vector $[3/5, 2/5]$ becomes an eigenvector of eigenvalue 1 for $M_{a,b}$, that is:

$$\begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix} \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}.$$

Multiplying out, equating sides and rewriting, this condition on (a, b) becomes equivalent to this linear system:

$$\begin{bmatrix} 3 & 2 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}.$$

Solving this as usual, we get that all solutions (a, b) are given by

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad s \in \mathbb{R}.$$

However, only those where both $0 \leq a \leq 1$ and $0 \leq b \leq 1$ are admissible. Clearly, looking at a this means we must restrict to $s \in [0, 1/2]$. Looking at b, we must restrict to $s \in [0, 1/3]$. Thus precisely the interval $s \in [0, 1/3]$ gives rise to Markov matrices with the desired property.

Inserting this back into our $M_{a,b}$, we see that the answer is

$$M = \begin{bmatrix} 1-2s & 3s \\ 2s & 1-3s \end{bmatrix}, \quad \text{where} \quad s \in [0, 1/3].$$

By the trace rule, the other eigenvalue is in each case $\lambda_2 = 1 - 5s$, so $-2/3 \leq \lambda_2 \leq 1$ (with $\lambda_2 = 1$ when $M = I$ only). Hence, picking an s we always get $\lim_{k \to +\infty} \lambda_2^k \to 0$ (unless $M = I$, where every vector is stationary), so the steady state indeed always arises.

Final sanity check: The example we started with appears as $s = 1/10$. $\square$