Solution Set 7, 18.06 Fall '11

1. Suppose $n > 1$. Prove that the determinant of an n by n matrix with every entry equal to 1 or -1 is even.

   Solution. Consider the "big sum" formula for computing the determinant. It will have $n!$ summands (there are $n!$ permutations), each one equal to ± 1 (each summand is some product of 1's and -1's). Since $n > 1$ the $n!$ will be an even number and we conclude by observing that an even number of ± 1's sums to an even number. □

2. Take the first matrix from problem 14 from 5.1. Calculate its determinant in at least 2 different ways that you've learned so far.

   Solution. (a) We can expand cofactors along the third row. This gives

$$(+1)(-1)\det\left(\begin{pmatrix} 2 & 3 & 0 \\ 6 & 6 & 1 \\ 2 & 0 & 7 \end{pmatrix}\right) + (-1)(3)\det\left(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 6 \\ 0 & 2 & 0 \end{pmatrix}\right) = (-1)[2(42) - 3(40)] - 3(0)$$
$$= 36.$$

   (b) We can row reduce. We can change the second row to $(0, 2, 0, 1)$ and the third row to $(0, 2, 3, 3)$ with the first row. Then we can change the third row to $0, 0, 3, 2)$ and the fourth row to $(0, 0, 0, 6)$ with the second row. This is upper triangular, so the determinant is $1 * 2 * 3 * 6 = 36$.

   (c) You can also use the big formula if you really want. There are 24 terms in the sum, though you can argue only about 10 of them contribute (thanks mostly to row 3). Since you are doing similar work with cofactor expansion, I'll omit the sum.

   □

3. Do problem 34 from 5.2.

   Solution. (a) The first two rows are dependent and so is the last three rows.

   (b) When you pick three entries from the last three rows, at least one of the entries should be zero since there are only two possibly non-zero columns.

   □

4. Do problem 20 from 5.3.

   Solution. You can row reduce and all, but you should also be on the lookout to use orthogonality. Take $H^T H$. This gives the matrix $4I$ (check it), which has determinant $4^4 = 256$. Since $\det(H) = \det(H^T)$, we must have $|\det(H)| = 16$. □