Solution Set 6, 18.06 Fall '11

1. Do problem 4 from 4.4.

   Solution. (a) The matrix $Q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has orthonormal columns but

   $$QQ^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

   (b) The vectors $\begin{bmatrix} 0 \end{bmatrix}$, $\begin{bmatrix} 0 \end{bmatrix}$ in $\mathbb{R}$ are orthogonal but are not linearly independent.

   (c) I claim that

   $$\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

   is such. These three vectors are clearly orthonormal. Therefore they are linearly independent (every set of pairwise orthogonal nonzero vectors is linearly independent - check this!). But any three linearly independent vectors in $\mathbb{R}^3$ form a basis and this verifies my claim. $\square$

2. Do problem 19 from 4.4.

   Solution. If $A = QR$ then $A^T A = R^T R = \boxed{\text{lower}}$ triangular times $\boxed{\text{upper}}$ triangular. Let c_1, c_2 denote the columns of A. Gram-Schmidt gives

   $$q_1' = c_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad q_2' = c_2 - \frac{\langle q_1', c_2 \rangle}{\langle q_1', q_1' \rangle} q_1' = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.$$

   Scaling to get unit lengths gives

   $$q_1 = \frac{q_1'}{\|q_1'\|} = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \quad q_2 = \frac{q_2'}{\|q_2'\|} = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

   Since

   $$R = \begin{bmatrix} \langle c_1, q_1 \rangle & \langle c_2, q_1 \rangle \\ 0 & \langle c_2, q_2 \rangle \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$$

   the desired $A = QR$ decomposition reads

   $$\begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}.$$ $\square$

3. Do problem 37 from 4.4. Hint: Find a vector in $c(A)$ that is orthogonal to $c(Q)$, then normalize.

Solution. The projection of a onto the column space of Q is $Pa = QQ^Ta$. So if you subtract $\boxed{QQ^Ta}$ and divide by $\boxed{\|a - QQ^Ta\|}$ you will get the new orthogonal vector $q = \frac{a-QQ^Ta}{\|a-QQ^Ta\|}$. This is of unit lenght and to check that q is orthogonal to the column space of Q we simply show that the projection of q onto $C(Q)$ is zero:

$$Pq = \frac{P(a - QQ^Ta)}{\|a - QQ^Ta\|} = \frac{QQ^T(a - QQ^Ta)}{\|a - QQ^Ta\|} = \frac{(QQ^Ta - Q(Q^TQ)Q^Ta)}{\|a - QQ^Ta\|} = 0. \qquad \square$$

4. Do problem 2 from 8.5.

Solution. To show that the corresponding functions are orthogonal we simply need to show that appropriate integrals vanish:

$$\int_{-1}^{1} 1 \cdot x \; dx = \frac{x^2}{2}\bigg|_{x=-1}^{1} = 0,$$

$$\int_{-1}^{1} 1 \cdot \left(x^2 - \frac{1}{3}\right) \; dx = \left(\frac{x^3}{3} - \frac{x}{3}\right)\bigg|_{x=-1}^{1} = 0,$$

$$\int_{-1}^{1} x \cdot \left(x^2 - \frac{1}{3}\right) \; dx = \left(\frac{x^4}{4} - \frac{x^2}{6}\right)\bigg|_{x=-1}^{1} = \frac{1}{4} - \frac{1}{6} - \left(\frac{1}{4} - \frac{1}{6}\right) = 0.$$

Writing $f(x) = 2x^2$ as a combination of those functions simply amounts to

$$f(x) = 2x^2 = 2\left(x^2 - \frac{1}{3}\right) + \frac{2}{3} \cdot 1. \qquad \square$$

5. Do problem 4 from 8.5.

Solution. Note that $x^3 - cx$ is perpendicular to 1 regardless of c:

$$\int_{-1}^{1} 1 \cdot (x^3 - cx) \; dx = \frac{x^4}{4} - \frac{cx^2}{2}\bigg|_{x=-1}^{1} = 0.$$

For $x^3 - cx$ to be perpendicular to x we must have

$$\int_{-1}^{1} x \cdot (x^3 - cx) \; dx = \left(\frac{x^5}{5} - \frac{cx^3}{3}\right)\bigg|_{x=-1}^{1} = \frac{1}{5} - \frac{c}{3} - \left(\frac{-1}{5} - \frac{-c}{3}\right) = 0,$$

i.e., $c = \frac{3}{5}$. It remains to show that with this c the function $x^3 - cx$ is also perpendicular to $x^2 - \frac{1}{3}$:

$$\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)\left(x^3 - \frac{3x}{5}\right) \; dx = \int_{-1}^{1} \left(x^5 - \frac{14}{15}x^3 + \frac{x}{5}\right) \; dx$$

$$= \left(\frac{x^6}{6} - \frac{14}{15}\frac{x^4}{4} + \frac{x^2}{10}\right)\bigg|_{x=-1}^{1} = 0,$$

where to obtain the last equality we have observed that the function in parentheses is even. $\square$

6. Do problem 12 from 8.5.

   Solution. The 5 by 5 "differentiation matrix" is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

   which succintly expresses the information about expressing the derivatives of the five functions in terms of those same functions:

$$\begin{aligned} 1' &= 0, \\ (\cos x)' &= -\sin x, \\ (\sin x)' &= \cos x, \\ (\cos 2x)' &= -2\sin 2x, \\ (\sin 2x)' &= 2\cos 2x. \quad \square \end{aligned}$$

7. (This problem is worth 20 points) In MATLAB or your favorite language, create $2n$-length discrete versions of $q_1 = 1/\sqrt{n}\cos(x)$ and $q_2 = 1/\sqrt{n}\cos(3x)$ by taking equal sized samples from $0 to 2\pi$, taking care to include 0 but exclude 2π. This means we want to think of each of these as column vectors $[x_0, \ldots, x_{2n-1}]^T$ where $x_i = i\pi/n$. In MATLAB this is `x=(0:(2*n-1))'*pi/n`. (before you go on, test to yourself that they're unit vectors). Let $Q = [q_1\ q_2]$.

   (a) Derive an identity for $\cos(3x)$ in terms of $\cos(x)$ (hint: you can use sum to product formulae). Use this identity to prove that $\cos(x)^3$ is in the span of $\cos(x)$ and $\cos(3x)$.

   Solution. Sure we can use sum to product formulae to express $\cos 3x = \cos(2x + x)$ in terms of trigonometric functions of arguments x and $2x$ and then use double angle formulas to get rid of all $\cos 2x$ and $\sin 2x$. But we can also use complex numbers to derive the identity in a much slicker way! Observe the Euler identity

$$e^{ix} = \cos x + i\sin x$$

   and then cube it. You will get

$$\begin{aligned} \cos 3x + i\sin 3x = e^{3ix} &= (\cos x + i\sin x)^3 \\ &= (\cos x)^3 - 3\cos x(\sin x)^2 + i(3(\cos x)^2\sin x - (\sin x)^3), \end{aligned}$$

   and consequently

$$\cos 3x = (\cos x)^3 - 3\cos x(\sin x)^2 = (\cos x)^3 - 3\cos x(1-(\cos x)^2) = 4(\cos x)^3 - 3\cos x.$$

   It is immediately clear that $(\cos x)^3$ lies in the span of $\cos 3x$ and $\cos x$ because

$$(\cos x)^3 = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x. \qquad \square$$

(b) Project $b = \cos(x)^3$ into the column space of Q as to obtain the best least squares fit (for a shortcut, see blue line under eq. 4 on page 233). This should give some expansion. Does b equal its projection? What does this have to do with the previous part of the problem (there should really only be one reasonable interpretation of this question)?

Solution. [See MATLAB code] $\square$

(c) Now project $b = \cos(x)^5$ onto the column space of Q. Does b equal its projection? If the answer is different from the previous part, why not?

Solution. [See MATLAB code] $\square$

8. Do problem 14 from 5.1.

Solution. As required, we do row operations:

$$\det\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix} = 1\cdot 2\cdot 3\cdot 6 = 36.$$

Similarly,

$$\det\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \det\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \det\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$= \det\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} = 2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4} = 5. \quad \square$$

9. Do problem 29 from 5.1.

Solution. Even though projection matrices $P = A(A^TA)^{-1}A^T$ are square, A appearing in the formula need not be. Therefore, it does not make sense to talk of $\det A$ and the "proof" breaks down. $\square$

MATLAB code

```
%%%%%%%%%%%%%%%%%%
%Problem 7 (b),(c)%
%%%%%%%%%%%%%%%%%%

n=10;
```

```
x=(0:(2*n-1))'*pi/n;
q1=cos(x)/sqrt(n);
q2=cos(3*x)/sqrt(n);
norm(q1)

ans =

     1

norm(q2)

ans =

     1


Q=[q1 q2];

b = (cos(x).^3)/sqrt(n);

% this shows projecting the cos^3 vector get
% itself back, since the norm of the difference
% is basically 0. You can also just
% display the two separately and look by eye.

norm((Q*(Q'*Q)^(-1)*Q'*b)-b)

ans =

   1.6059e-16

% now we do the cos^5 vector. Here the difference
% is very far from 0

c = (cos(x).^5)/sqrt(n);

norm((Q*(Q'*Q)^(-1)*Q'*c)-c)

ans =

    0.0625

% an additional thing you can do to check the
% coefficients:
% cos(3x)=4*(cos(x))^3-3*cos(x),
```

```
% hence cos(x)^3 = (1/4)cos(3x)+(3/4)cos(x)
% the following shows this:

Q'*b

ans =

    0.7500
    0.2500


diary off
```