Solution Set 5, 18.06 Fall '11

1. Take two connected graphs A and B with a and b vertices respectively. Let their union be C (i.e. a big graph with A and B as two disjoint parts).

   (a) What is the rank of C's adjacency matrix?

   Solution. If R and S are the adjacency matrices of A and B then the adjacency matrix of C is the block matrix

$$C = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}.$$

   We see immediately that the rank of C is the sum of the ranks of A and B: this is easy to see, for instance, if you think about how elimination would proceed for C and what the pivot columns would be. The rank of R is $a-1$ because it is an adjacency matrix of a connected graph with a vertices; similarly, the rank of S is $b-1$. Therefore, the rank of C is $a+b-2$. You might wonder: shouldn't the rank of C be $a+b-1$ because it's an adjanceny matrix of a graph? No, because it's an adjacency matrix for a graph which is not connected! □

   (b) What is its nullspace?

   Solution. The dimension of the nullspace is $(a+b)-(a+b-2)=2$. Since

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

   (the first vector has a ones and b zeroes, the second one has a zeroes and b ones) are both in the nullspace of C and are also linearly independent (because $a, b > 0$; the empty graph is not connected!), they form a basis of the nullspace. I.e., the nullspace consists of all vectors whose first a coordinates are equal and whose last b coordinates are equal. □

2. Free points!!!

   Solution. Yay! □

3. Do problem 21 from 4.1.

Solution. Such vectors are, for instance,

$$\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -1 \\ -1 \end{bmatrix}.$$

They are easily seen to be in $S^{\perp}$, independent, and as the dimension of $S^{\perp}$ is $4-2=2$ they must also form a basis for $S^{\perp}$, so in particular span. Finding $S^{\perp}$ is the same as solving $Ax = 0$ for $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$ because the two equations in $Ax = 0$ are saying precisely that x is orthogonal to S. □

4. Do problem 28 from 4.1.

Solution. (a) Perpendicularity of $(1, 1, 1)$ and $(1, 1, -2)$ only shows that the two planes contain lines that are orthogonal to each other. A plane is two-dimensional, so to check orthogonality we would have to extend those two vectors to bases and then check that each vector in the basis for the first plane is perpendicular to each vector in the basis for the second plane.

(b) The subspace spanned by $(1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1)$ is two-dimensional. So is the subspace spanned by $(1, -1, 0, 0, 0)$ and $(2, -2, 3, 4, -4)$. In a 5-dimensional space, however, the dimensions of any pair of orthogonal complements must sum up to 5.

(c) Two distinct lines in a plane meet only in the zero vector but they are not necessarily orthogonal. □

5. Do problem 11(a) from 4.2. Check the "it should..." part!

Solution. First we compute A^TA and A^Tb:

$$A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad A^Tb = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

Now we have to solve

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \widehat{x} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

As $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is invertible there is a unique solution, namely, $\widehat{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ (easy guesswork, but we could've as well found A^{-1} and computed $\widehat{x} = A^{-1}b$). Finally we compute the projection p:

$$p = A\widehat{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}.$$

Let us check that $e = b - p = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$ is perpendicular to the columns of A. But this is evident because the columns of A have last coordinates zero. □

6. Do problem 17 from 4.2. Briefly explain your answer.

   Solution. Indeed, this follows from the laws for operations with matrices:

   $$(I - P)^2 = (I - P)(I - P) = I - P - P + P^2 = I - P,$$

   because $P^2 = P$. Therefore, both P and $I - P$ are projection matrices.

   When P projects onto the column space of A, $I - P$ projects onto the orthogonal complement of the column space of A, namely, onto the left nullspace of A. Indeed, the images of $I - P$ and P are perpendicular, because $(I - P)P = P(I - P) = 0$ and they span because any x can be decomposed as $x = (I - P)x + Px$. Therefore, they are orthogonal complements. □

7. Do problem 22 from 4.2.

   Solution. To show that a matrix P is symmetric we only need to show that $P^T = P$. For our P this is an easy calculation:

   $$P^T = (A(A^TA)^{-1}A^T)^T = (A^T)^T((A^TA)^T)^{-1}A^T = A(A^TA)^{-1}A^T = P.$$ □

8. On a computer:

   (a) project a vector b onto the column space of a matrix A with independent columns (in MATLAB, try `A=randn(3,2)`. When making the projection matrix, keep in mind that `A'` is the transpose and `\inv(A)` is the inverse). Project the result onto the same A. Explain the results.

   Solution. [See MATLAB code] □

   (b) Construct an A where the columns are not independent. Now try to make a projection matrix. What happens? Explain why MATLAB would do this.

   Solution. [See MATLAB code] □

   (c) Let the projection matrix you get in the first part be P. Add $P + P^2 + \cdots + P^{100}$. (in MATLAB this would be the following code: `Z=0; for i=1:100, Z=Z+P\^i; end`. Compare the answer to P. Explain what you see.

   Solution. [See MATLAB code] □

9. Do problem 6 from 4.3.

Solution. First we compute the inner products $a^T b$ and $a^T a$:

$$a^T b = 1 \cdot 0 + 1 \cdot 8 + 1 \cdot 8 + 1 \cdot 20 = 36,$$
$$a^T a = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 4.$$

Then

$$\widehat{x} = a^T b / a^T a = 36/4 = 9.$$

The projection p equals

$$p = \widehat{x} a = 9 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix}.$$

To check that

$$e = b - p = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix} = \begin{bmatrix} -9 \\ -1 \\ -1 \\ 11 \end{bmatrix}$$

is perpendicular to a we compute the dot product

$$e^T a = -9 \cdot 1 + (-1) \cdot 1 + (-1) \cdot 1 + 11 \cdot 1 = 0.$$

The shortest distance $\|e\|$ from b to the line through a is

$$\|e\| = \sqrt{(-9)^2 + (-1)^2 + (-1)^2 + 11^2} = \sqrt{204} = 2\sqrt{51}.$$ □

10. Do problem 12 from 4.3.

Solution. (a) The inner product $a^T a$ is $a^T a = 1 + \cdots + 1 = m$. The inner product $a^T b$ is $a^T b = b_1 + \cdots + b_m$. Therefore, the solution $\widehat{x}$ to $a^T a \widehat{x} = a^T b$ is

$$\widehat{x} = \frac{b_1 + \cdots + b_m}{m},$$

the mean of the b_i's.

(b) We calculate

$$e = b - a\widehat{x} = \begin{bmatrix} b_1 - \frac{b_1 + \cdots + b_m}{m} \\ \vdots \\ b_m - \frac{b_1 + \cdots + b_m}{m} \end{bmatrix}.$$

The variance is

$$\begin{aligned} \|e\|^2 &= \left(b_1 - \frac{b_1 + \cdots + b_m}{m}\right)^2 + \cdots + \left(b_m - \frac{b_1 + \cdots + b_m}{m}\right)^2 \\ &= b_1^2 + \cdots + b_m^2 - \frac{2}{m}(b_1 + \cdots + b_m)(b_1 + \cdots + b_m) + \frac{(b_1 + \cdots + b_m)^2}{m} \\ &= b_1^2 + \cdots + b_m^2 - \frac{1}{m}(b_1 + \cdots + b_m)^2. \end{aligned}$$

The standard deviation is

$$\|e\| = \sqrt{b_1^2 + \cdots + b_m^2 - \frac{1}{m}(b_1 + \cdots + b_m)^2}.$$

(c) First we find $e = b - \widehat{b} = \begin{bmatrix}1\\2\\6\end{bmatrix} - \begin{bmatrix}3\\3\\3\end{bmatrix} = \begin{bmatrix}-2\\-1\\3\end{bmatrix}$. It is evident that e is perpendicular to p because the dot product vanishes: $(-2)\cdot 3 + (-1)\cdot 3 + 3\cdot 3 = 0$. The projection matrix P is given by the formula

$$P = \frac{aa^T}{a^Ta} = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}.$$

Indeed, one readily verifies that $P^2 = P$ and also

$$Pb = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\6\end{bmatrix} = \begin{bmatrix}3\\3\\3\end{bmatrix} = p.$$

□

MATLAB code

```
%%%%%%%%%%%%%
% Problem 8 %
%%%%%%%%%%%%%

A=randn(3,2);b=[1 1 1]';

P=A*inv(A'*A)*A';

P*b

ans =

  -0.144525932578878

   0.151266288307688

   0.038512058052834

P*P*b

ans =

  -0.144525932578878
```

```
   0.151266288307689

   0.038512058052834

% P^2=P for a projection, so P*P*b = P*b.

%(b)

A=randn(3,1)*randn(1,2); %% What is the rank of this matrix?

inv(A'*A)

{Warning: Matrix is close to singular or badly scaled.

         Results may be inaccurate. RCOND = 1.596964e-017.}

ans =

  1.0e+014 *

  -0.668731211789606  -1.371973404495029

  -1.371973404495029  -2.814749767106560

A*inv(A'*A)*A'

{Warning: Matrix is close to singular or badly scaled.

         Results may be inaccurate. RCOND = 1.596964e-017.}

ans =

  -0.000471959154735   0.000372942179054   0.003895037519367

  -0.008909336176797   0.007040158484826   0.073527970234564

   0.003775673237879  -0.002983537432432  -0.031160300154939

% It seems we have an error. The reason is that
% we don't have full rank.

%(c)

Z=0; for i=1:100, Z=Z+P^i; end
```

```
P

P =

   0.556666154342449  -0.328758282912762  -0.372433804008565

  -0.328758282912762   0.756206277408302  -0.276181706187851

  -0.372433804008565  -0.276181706187851   0.687127568249249

Z

Z =

  55.666615434245969 -32.875828291277372 -37.243380400856793

 -32.875828291277287  75.620627740831850 -27.618170618785278

 -37.243380400856836 -27.618170618785214  68.712756824925364

% Reason:
% Z is 100*P as P^i=P for all 100 instances.
```