Solution Set 4, 18.06 Fall '11

1. Do Problem 1 from 3.6.

   (a) *Solution.* Rank equals both the dimension of the column space and the dimension of the row space:
   $$\dim C(A) = 5, \quad \dim C(A^T) = 5.$$
   Now we can easily figure out the dimensions of the nullspace and of left nullspace:
   $$\dim N(A) = 9 - 5 = 4, \quad \dim N(A^T) = 7 - 5 = 2.$$
   The sum of all four dimensions is
   $$5 + 5 + 4 + 2 = 16 = 7 + 9.$$
   □

   (b) *Solution.* The left nullspace must have dimension $3 - 3 = 0$ and therefore is trivial, $N(A^T) = 0$. The column space is of dimension 3 and is a subspace of $\mathbb{R}^3$ which means that $C(A) = \mathbb{R}^3$. □

2. Do Problem 3 from 3.6.

   Solution. We are given that the echelon form of A is
   $$U = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
   The pivot variables are x_2, x_4 and therefore the corresponding columns of A form a basis
   $$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$
   for the column space of A.

   A basis for the nullspace will be the three special solutions to $Ax = 0$ obtained by setting one of the free variables to 1, the others to 0:
   $$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$
   Row operations do not change the row space and therefore pivot rows will form a basis for the row space:
   $$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

The left nullspace has dimension $3-2=1$ so any nonzero element of the left nullspace will be its basis, for instance

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}. \qquad \square$$

3. Do problem 9 from 3.6.

Solution. Firstly, all of those matrices have the same rank 0 or 1 (dependent on whether A is zero or not). Indeed, the case $A=0$ being clear, we note that in the case $A \neq 0$ the column space of any of those matrices will be one dimensional spanned by the vector $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ (the exact number of coordinates varies dependent on the number of rows in a matrix in question).

(a)
- The column spaces cannot be the same because they are subspaces of different spaces.
- The nullspaces are the same: the zero subspace of $\mathbb{R}^1$ if $A \neq 0$, all of $\mathbb{R}^1$ if $A=0$.
- The row spaces are the same: equal to the zero subspace of $\mathbb{R}^1$ if $A=0$, equal to all of $\mathbb{R}^1$ if $A \neq 0$.
- The left nullspaces cannot be the same because they are subspaces of different spaces.

(b)
- The column spaces are the same: equal to the zero subspace of $\mathbb{R}^2$ if $A=0$, equal to the line in $\mathbb{R}^2$ spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ otherwise.
- The nullspaces cannot be the same because they are subspaces of different spaces.
- The row spaces cannot be the same because they are subspaces of different spaces.
- The left nullspaces are the same: equal to the line in $\mathbb{R}^2$ spanned by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ if $A \neq 0$, equal to all of $\mathbb{R}^2$ if $A=0$. $\square$

4. Do Problem 25 from 3.6.

Solution. (a) True. Both the number of pivot rows and the number of pivot columns are equal to the rank of the matrix.

(b) False. $A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has $\mathbb{R}^2$ for its left nullspace, whereas $A^T = \begin{bmatrix} 0 & 0 \end{bmatrix}$ has $\mathbb{R}^1$ for its left nullspace.

(c) False. Both the row space and the column space of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are equal to $\mathbb{R}^2$ but

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = A^T.$$

(d) True. The row space of A is the column space of A^T which in our case is also the column space of $-A$. But A and $-A$ have the same column space (the columns of $-A$ are nonzero scalar multiples of the columns of A). □

5. Do Problem 27 from 3.6.

Solution. I would choose $d = \frac{cb}{a}$ because then the second column would be a scalar multiple of the first and the column space would be 1-dimensional (because $a \neq 0$).

The dimension of the row space is equal to the rank which is 1. The dimension of the nullspace is $2 - 1 = 1$. For a 1-dimensional vector space any nonzero vector is a basis. In particular, $\begin{bmatrix} a \\ b \end{bmatrix}$ is a basis for the row space and $\begin{bmatrix} -b \\ a \end{bmatrix}$ is a basis for the nullspace.

Any element of the row space is of the form $\begin{bmatrix} ax \\ bx \end{bmatrix}$. Any element of the nullspace is of the form $\begin{bmatrix} -by \\ ay \end{bmatrix}$. Therefore the dot product of an element of the row space and an element of the nullspace is $ax \cdot (-by) + bx \cdot ay = 0$, i.e., the row space and the nullspace are perpendicular! □

6. Do problem 28 from 3.6 (challenge problem not required).

Solution. It is easy to see that

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for the row space of B. Therefore, the dimension of the row space of B is 2. This means that the dimension of the left nullspace of B is $8 - 2 = 6$. It is easy to verify that the following 6 vectors are both in the left nullspace of B and are linearly independent and therefore form a desired basis:

$$\begin{bmatrix}1\\0\\-1\\0\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\\-1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\\0\\0\\-1\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\-1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\\0\\-1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\-1\end{bmatrix}.$$

It is easy to see that

$$v_1=\begin{bmatrix}r\\n\\b\\q\\k\\b\\n\\r\end{bmatrix},v_2=\begin{bmatrix}p\\p\\p\\p\\p\\p\\p\\p\end{bmatrix}$$

span the row space of C. Also, because r,n,b,k,q,p are all different, these two vectors form a basis for the row space unless $p=0$ in which case v_1 alone forms a desired basis. In the case where $p\neq 0$ the dimension of the left nullspace is $8-2=6$ and the following 6 independent vectors of the left nullspace form a basis for it:

$$\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\\0\\0\\-1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}.$$

In the case where $p=0$ the left nullspace is $8-1=7$-dimensional and the following linearly independent vectors form a basis for it:

$$\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix}.$$

□

7. Do problem 2 from 8.2.

Solution. First we write down A remembering that the rows correspond to edges whereas the columns correspond to vertices:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

A vector in the nullspace of $A^T = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ is, for instance, $y = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. For this y the current going around the triangle is 0. □

8. Do problem 17 from 8.2.

Solution. (a) For any incidence matrix the number of independent columns is always one less than the total number of columns. In our case there are $9 - 1 = 8$ independent columns.

(b) $A^T y = f$ has a solution if and only if f belongs to the row space of A. The row space of any incidence matrix consists of all vectors orthogonal to $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. If $f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$ this condition is the same as saying that $f_1 + \cdots + f_n = 0$.

(c) Since each edge contributes twice to the sum (once for each endpoint) the sum of the diagonal entries is twice the number of edges, i.e., $= 2 \cdot 12 = 24$. □

9. Take the 8 vertices and the 12 edges of a cube, and look at its incidence matrix.

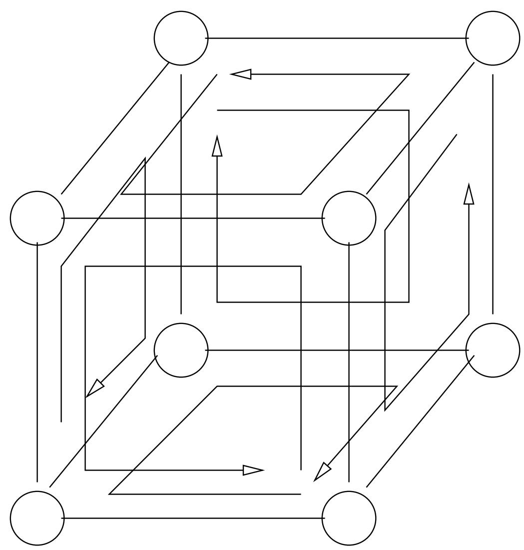
(a) Find the dimensions of the four fundamental subspaces of the graph.

Solution.
- The nullspace of an incidence matrix is always 1-dimensional.
- The dimension of the column space of an incidence matrix is always one less than the total number of columns. In our case the dimension of the column space is $8 - 1 = 7$.
- The dimension of the row space is equal to the dimension of the column space is equal to the rank for any matrix. In our case the dimension of the row space is 7.
- The dimension of the left nullspace is $12 - 7 = 5$ because there are 12 rows (corresponding to the edges), whereas the row space is 7-dimensional. □

(b) Show that the six "loops" around the faces of a cube are linearly dependent; be clear about what vector space these loops live in. You're allowed to just draw a picture to show this, but explain what your picture means.

Solution. Indeed, each loop is comprised of four edges which live in the row space of the matrix because rows correspond to edges. If the edges on the loop are oriented so that they form a cycle then the sum of the rows corresponding to these four edges is zero because each vertex on the loop is an endpoint to one edge and an initial point of one edge (the corresponding coordinate in the sum has two nonzero summands: one 1 and one -1, whereas a coordinate corresponding to a vertex not on the loop has all corresponding summands 0). In general each loop will not necessarily be oriented this way but one can multiply each edge of the loop by ± 1 (reverse the orientation of some edges) to reduce to the situation where it is, in which case the reasoning above applies.

An alternate way to see the situation is via the picture below: suppose we have picked some edge orientations. Then take each loop (again, we think of the loop as a linear combination of 4 edges) and put a coefficient on the corresponding edge in the loop that is 1 if the cycle drawn in the picture goes "along" with the edge and -1 otherwise. If we now add these 6 loops, each edge appears twice, one with a 1 and one with a -1 (since in the picture each edge has two loops going in the opposite direction) so they must cancel.



□

10. Do problem 3 from 4.1.

Solution. (a) $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$ is such a matrix.

(b) Any vector in the nullspace is orthogonal to any vector in the row space. But $2 \cdot 1 + (-3) \cdot 1 + 5 \cdot 1 \neq 0$.

(c) $A^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is saying that the first column of A^T is zero. This means that the first row of A is zero. But then Ax has its first entry 0 for any x. This means that $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is impossible showing that there is no such matrix A.

(d) The matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has this property.

(e) This is impossible because the sum of the sums of the columns of A must be equal to the sum of the sums of the rows of A (both are equal to the sum of the entries of A). $\square$