18.06 Professor Edelman Quiz 3 December 5, 2011

Grading

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Your PRINTED name is:________________________

Please circle your recitation:

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1 (24 pts.)

Let $A = \begin{pmatrix} .5 & 0 & 0 \\ .5 & .9 & 0 \\ 0 & .1 & 1 \end{pmatrix}$.

1. (4 pts) True or False: The matrix A is Markov.

   True. Markov matrices have columns that sum to 1 and have non-negative entries. The answer of false applies to what is known as "Positive Markov Matrices."

2. (6 pts) Find a vector $x \neq 0$ and a scalar λ such that $A^T x = \lambda x$.

   The obvious choice is (1,1,1) with λ =1 as this is the column sum property. Also easy to see is (1,0,0) with $\lambda = 0.5$.

3. (4 pts) True or False: The matrix A is diagonalizable. (Explain briefly.)

True. The three eigenvalues, on the diagonal, are distinct.

4. (4 pts) True or False: One singular value of A is $\sigma = 0$. (Explain briefly.)

False. The matrix is nonsingular, since it has no zero eigenvalues. Nonsingular square matrices have all n singular values positive.

5. (6 pts) Find the three diagonal entries of e^{At} as functions of t.

They are $e^{t}, e^{0.5t}, e^{0.9t}$.

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2 (30 pts.)

1. (5 pts) An orthogonal matrix Q satisfies $Q^TQ = QQ^T = I$. What are the n singular values of Q?

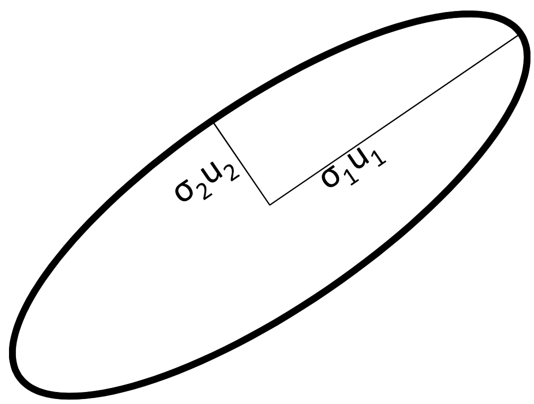
They are all 1. The singular values are the positive square roots of the eigenvalues of $QQ^T = Q^TQ = I$.

2. (10 pts) Let $A = \begin{pmatrix} 1 & & \\ & -2 & \\ & & 3 \end{pmatrix}$. Find an SVD, meaning $A = U\Sigma V^T$, where U and V are orthogonal, and $\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}$ is diagonal with $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$. (Be sure that the factorization is correct and satisifies all stated requirements.)

$A = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} \begin{pmatrix} 3 & & \\ & 2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$. The singular values are in decreasing order and are positive. One can compute AA^T and A^TA, but easier to rig the permutation matrices and correct the sign.

3. (15 pts) The 2×2 matrix $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$, where $\sigma_1 > \sigma_2 > 0$ and both u_1, u_2 and v_1, v_2 are orthonormal bases for R^2.

The set of all vectors x with $\|x\| = 1$ describes a circle in the plane. What shape best describes the set of all vectors Ax with $\|x\| = 1$? Draw a general picture of that set labeling all the relevant quantities σ_1, σ_2, u_1, u_2 and v_1, v_2. (Hint: Why are u_1, u_2 relevant and v_1, v_2 not relevant?)



The svd rotates (or reflects) the circle with V^T, scales to an ellipse with axes in the coordinate directions through Σ, and then a rotated ellipse with axes in the direction $u_1$and u_2 after U is applied. The Σ scales the x and y axes by $\sigma_1$and σ_2 respectively, and σ_1 is the longer of the two.

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3 (16 pts.)

1. (6 pts) Let $x \neq 0$ be a vector in R^3. How many eigenvalues of $A = xx^T$ are positive? zero? negative? (Explain your answer.) (Hint: What is the rank?)

A is symmetric pos semidefinite and rank 1, so there are 1 positive, 2 zero, and no negative eigenvalues.

2. (6 pts) a) What are the possible eigenvalues of a projection matrix?

0 and 1 (Since $P^2 = P$, $\lambda^2 = \lambda$.)

b) True or False: every projection matrix is diagonalizable.

True, every projection matrix is symmetric, hence diagonalizable.

3. (4 pts) True or False: If every eigenvalue of A is 0, then A is similar to the zero matrix.

False. A Jordan block with zero eigenvalues is not similar to the zero matrix for $n > 1$.

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4 (30 pts.)

Consider the matrix $A = \begin{pmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ with parameter x in the (1,1) position.

1. (10 pts) Specify all numbers x, if any, for which A is positive definite. (Explain briefly.)

   > No x, the matrix is clearly singular with two equal rows and two equal columns.

2. (10 pts) Specify all numbers x, if any, for which e^A is positive definite. (Explain briefly.)

   > The eigenvalues of e^A are the exponentials of the eigenvalues of the matrix A. Since A is symmetric the eigenvalues are real, and thus exponentials are positive. A symmetric matrix with positive eigenvalues is positive definite.

3. (10 pts) Find an x, if any, for which $4I - A$ is positive definite. (Explain briefly.)

One can take any $x < 3$. The easiest choice is $x = 1$. With this guess the matrix has two eigenvalues 0 and one eigenvalue 3 both less than 4, so $4 - \lambda > 0$ for all three eigenvalues. Systematically, one can consider the three upper left determinants of $4I - A$ which are $4 - x$, $11 - 3x$, and $24 - 8x$. They are all positive if and only if $x < 3$.

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