18.06 Professor Edelman Quiz 2 November 4, 2011

Grading

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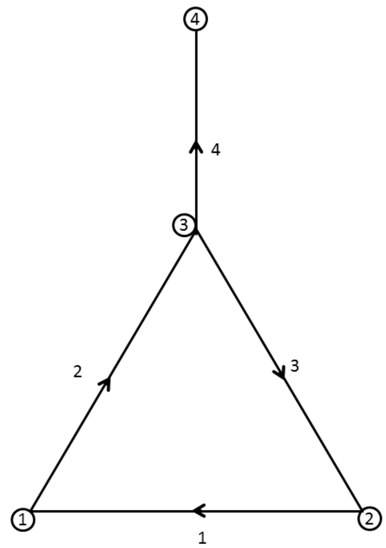
Your PRINTED name is:________________________

Please circle your recitation:

1	T 9	2-132	Kestutis Cesnavicius	2-089	2-1195	`kestutis`
2	T 10	2-132	Niels Moeller	2-588	3-4110	`moller`
3	T 10	2-146	Kestutis Cesnavicius	2-089	2-1195	`kestutis`
4	T 11	2-132	Niels Moeller	2-588	3-4110	`moller`
5	T 12	2-132	Yan Zhang	2-487	3-4083	`yanzhang`
6	T 1	2-132	Taedong Yun	2-342	3-7578	`tedyun`

1 (30 pts.)

Consider the directed graph with four vertices and four edges pictured below:



1. (7 pts) The 4×4 incidence matrix (following class conventions) of this directed graph is:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

2. (7 pts) Find the determinant of the incidence matrix. (The easy way or the hard way)

det(A)=0. This can be done by direct expansion or appeal conceptually to show matrix is not invertible. Can use that the sum of all columns is 0, that we know from book/class that rank is $4 - 1 = 3$, or that the nullspace includes $(1, 1, 1, 1)^T$ (note this is equivalent to the columns summing to 0 condition) and thus is nonempty.

3. (8 pts) Find a basis for the column space of the incidence matrix (Note this can be done with or without the answer in part 1.)

We need $4-1=3$ basis vectors. Any three columns of A form a basis, as would any three independent vectors whose first three components sum to 0.

4. (8 pts) Consider whether or not it is possible to have an incidence matrix for a graph with n nodes and n edges that is invertible. If it is possible, draw the directed graph, if not possible, argue briefly why not.

Impossible, as the ones vector is in the nullspace of every incidence matrix for every graph. As in problem 2, can also argue from book/class knowledge, or explicitly show that the column sums are 0. You can also show that the rows corresponding to a loop must sum to 0 and we must have a loop.

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2 (20 pts.)

1. (10 pts) Project the function $\sin(x) + \cos(x)$ defined on the interval $[0, 2\pi]$ onto the three dimensional space of functions spanned by $\cos x$, $\cos 2x$, and $\cos 3x$. Express the (hint: very simple) answer in simplest form. Briefly explain your answer.

   The projection is $\cos x$. We know $\sin x$ is orthogonal to the space and projects to 0, while $\cos x$ is already in the space.

2. (10 pts) Write down all $n \times n$ permutation matrices that are also projection matrices. (Explain briefly.)

   Since $P^2 = P$, multiplying both sides by P^{-1}, we get $P = I$ is the only projection, permutation. Note P^{-1} exists and you need it to exist; it is P^T, or you can note that it has nonzero determinant and is thus invertible.

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3 (15 pts.)

1. (10 pts) What are all possible values for the determinant of a projection matrix? (Please explain briefly.)

Since $P^2 = P$, $\det(P)^2 = \det(P)$so that only 0 or 1 are possible.

2. (5 pts) What are all possible values for the determinant of a permutation matrix? (Please explain briefly.)

Starting with I, a permutation matrix is obtained through row exchanges, therefore we can get only ± 1.

4 (35 pts.)

1. (20 pts) The matrix A is 2000×2000 and $A^T A = I$. Let v be the vector $[1, 2, 3, \ldots, 2000]^T$. Let v_1 be the projection of v onto the space spanned by the first 1000 columns of A. Let v_2 be the projection of v onto the space spanned by the remaining 1000 columns of A. What is $v_1 + v_2$ in simplest form? Why? Give an example of a 2000×2000 A, where $A^T A \neq I$, and where $v_1 + v_2$ gives a different answer.

   $v_1 + v_2 = v$ since projections onto orthogonal complements add to the identity. Here's something more explicit: let A take block form $\begin{bmatrix} A_1 & A_2 \end{bmatrix}$. Then v_1 and v_2 are projections onto the column spaces of A_1 and A_2 respectively. Note that since $A^T A = I$, we have:

   $$I = AA^T = A_1 A_1^T + A_2 A_2^T.$$

   Adding the two projections (recall in the orthogonal case this is just $(A_1 A_1^T v + A_2 A_2^T v)$ gives $Iv = v$ by the above equation.
   An easy example where we get a different answer is if A is the zero matrix, where we have $v_1 + v_2 = 0$ always for every v.

2. (15 pts) In a matrix A, (which may not be invertible) the cofactors from the first row are $C_{11}, C_{12}, \ldots, C_{1n}$.. Prove that the vector $C = (C_{11}, C_{12}, \ldots, C_{1n.})$ is orthogonal to every row of A from row 2 to row n. Hint: the dot product of C with row i ($i = 2, \ldots, n$) is the determinant of what matrix?

   Take the matrix A and replace row 1 with row i. This matrix has two equal rows hence 0 determinant. The determinant expansion by cofactors is the desired dot product.

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