18.06 Prof. Edelman Final Exam December 22, 2011

Grading

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Your PRINTED name is: ____________________

Please circle your recitation:

1	T 9	2-132	Kestutis Cesnavicius	2-089	2-1195	kestutis
2	T 10	2-132	Niels Moeller	2-588	3-4110	moller
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1 (13 pts.)

Suppose the matrix A is the product

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) (3 pts.) What is the rank of A?

(b) (5 pts.) Give a basis for the nullspace of A.

(c) (5 pts.) For what values of t (if any) are there solutions to $Ax = \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}$?

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2 (12 pts.)

Let $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$.

(a) (3 pts.) Find a basis for the column space of A.

(b) (3 pts.) Find a basis for the column space of Σ where $A = U\Sigma V^T$ is the SVD of A.

(c) (3 pts.) Find a basis for the column space of the matrix exponential e^A.

(d) (3 pts.) Find a non-zero constant solution (meaning no dependence on t) to $\frac{d}{dt}u(t) = Au(t)$.

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3 (12 pts.)

(a) (3 pts.) Give an example of a nondiagonalizable matrix A which satisfies $\det(A - tI) = (4-t)^4$

(b) (3 pts.) Give an example of two different matrices that are similar and both satisfy $\det(A - tI) = (1-t)(2-t)(3-t)(4-t)$.

(c) (3 pts.) Give an example if possible of two matrices that are **not** similar that satisfy $\det(A - tI) = (1-t)(2-t)(3-t)(4-t)$.

(d) (3 pts.) Give an example of two different 4 by 4 matrices that have singular values 4, 3, 2, 1.

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4 (16 pts.)

The matrix $G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$.

(a) (3 pts.) This matrix has two eigenvalues $\lambda = 2$, and one eigenvalue $\lambda = -2$. Given that, find the fourth eigenvalue.

(b) (5 pts.) Find a real eigenvector and show that it is indeed an eigenvector.

(Problem 4 continued.) The matrix $G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$.

(c) (4 pts.) Is G a Hermitian matrix? Why or why not. (Remember Hermitian means that $H_{jk} = \bar{H}_{kj}$ where the bar indicates complex conjugate.)

(d) (4 pts.) Give an example of a real non-diagonal matrix X for which $G^H X G$ is Hermitian.

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5 (16 pts.)

The following operators apply to differentiable functions $f(x)$ transforming them to another function $g(x)$. For each one state clearly whether it is linear or not (explanations not needed). (2 pts each problem)

(a) $g(x) = \frac{d}{dx} f(x)$

(b) $g(x) = \frac{d}{dx} f(x) + 2$

(c) $g(x) = \frac{d}{dx} f(2x)$

(d) $g(x) = f(x+2)$

(e) $g(x) = f(x)^2$

(f) $g(x) = f(x^2)$

(g) $g(x) = 0$

(h) $g(x) = f(x) + f(2)$

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6 (20 pts.)

Let $A = I_3 - cE_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - c \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(a) (4 pts.) There are two values of c that make A a projection matrix. Find them by guessing, calculating, or understanding projection matrices. Check that A is a projection matrix for these two c.

(b) (4 pts.) There are two values of c that make A an orthogonal matrix. Find them and check that A is orthogonal for these two c.

(c) (4 pts.) For which values of c, if any, is A diagonalizable?

(Problem 6 Continued) Let $A = I_3 - cE_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - c \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(d) (4 pts.) Find the eigenvalues of A^{-1} (if it exists) in terms of c. (Hint: find the eigenvalues of E_3 first.)

(e) (4 pts.) For which values of c, if any, is A positive definite?

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7 (11 pts.)

The general equation of a circle in the plane has the form $x^2 + y^2 + Cx + Dy + E = 0$. Suppose you are trying to fit $n \geq 3$ distinct points $p_i = (x_i, y_i)$, $i = 1, ..., n$ to obtain a "best" least squares circle, it is reasonable to write a generally unsolvable equation

$$A \begin{pmatrix} C \\ D \\ E \end{pmatrix} = b$$

for the coefficients C, D, and E.

(a) (7 pts.) Describe A and b clearly, indicating the number of rows and columns of A and the number of elements in b.

(b) (4 pts.) When $n = 3$ it is possible to describe when the equation is and is not solvable. You can use your geometric intuition or a determinant area formula to describe the condition on the points p_1, p_2, p_3 that makes A singular. Give a simple geometrical description of this condition. (We are looking for a specific word – so only a short answer will be accepted.)

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Linear Algebra is really really useful. Hope you enjoyed the class and find an opportunity to use the ideas you learned in new situations. Thanks for taking the class, have a great holiday, and wishing you all a happy 2012!