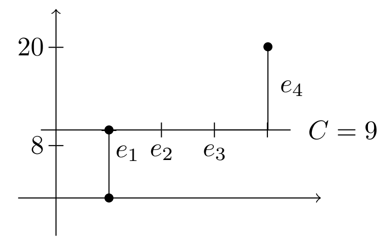
18.06 F10 Problem Set 6 Solutions Due Thursday, 21 Oct.

1. Problem 5, section 4.3, p.226.

Solution: In matrix form, the unsolvable equations become $A\hat{x} = b$ with $\boxed{A = [1; 1; 1; 1]}$ and $b = [0; 8; 8; 20]$. So $A^{\mathrm{T}}A\hat{x} = A^{\mathrm{T}}b$ is $4C = 36$. Thus the best height C is given by $\boxed{C = 9}$ and the error vector $e = b - A\hat{x}$ by $\boxed{e = [-9; -1; -1; 11]}$.
The pictorial form of the horizontal line and the four errors is drawn on the right.



2. Problem 12, section 4.3, p.228.

Solution:

(a) Here $a^{\mathrm{T}}a = m$ and $a^{\mathrm{T}}b = b_1 + \cdots + b_m$. So $a^{\mathrm{T}}a\hat{x} = a^{\mathrm{T}}b$ yields the mean: $\boxed{\hat{x} = (b_1 + \cdots + b_m)/m}$.

(b) Here $e = b - a\hat{x}$ is $e = [b_1 - \hat{x}; \ldots ; b_m - \hat{x}]$. So the variance is $\boxed{||e||^2 = (b_1 - \hat{x})^2 + \cdots + (b_m - \hat{x})^2}$ and the standard deviation is $\boxed{||e|| = \sqrt{(b_1 - \hat{x})^2 + \cdots + (b_m - \hat{x})^2}}$.

(c) Here $p = [3; 3; 3]$ and $e = [-2; -1; 3]$. So $p^{\perp}e = 3*(-2) + 3*(-1) + 3*3 = 0$ and $P = a(a^{\mathrm{T}}a)^{-1}a^{\mathrm{T}} = \boxed{\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$.

3. Problem 2.5, section 4.3, p.229.

Solution: Geometrically, the condition is that the segment from the first point to the second has the same slope as the segment from the second point to the third; that is,

$$\boxed{(b_2 - b_1)/(t_2 - t_1) = (b_3 - b_2)/(t_3 - t_2)}.$$

Algebraically, the condition is that (t_1, b_1) and (t_2, b_2) and (t_3, b_3) must satisfy some linear equation $C + Dt = b$. In other words, the vector $[b_1; b_2; b_3]$ must be in column space of the matrix $A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix}$. That space is the orthogonal complement of the left nullspace $N(A^{\mathrm{T}})$. To find $N(A^{\mathrm{T}})$, we row reduce A^{T} all the way to echelon form $\mathrm{rref}(A^{\mathrm{T}})$:

$$A^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & t_2 - t_1 & t_3 - t_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & (t_3 - t_1)/(t_2 - t_1) \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & (t_2 - t_3)/(t_2 - t_1) \\ 0 & 1 & (t_3 - t_1)/(t_2 - t_1) \end{bmatrix} = \mathrm{rref}(A^{\mathrm{T}}).$$

Hence $N(A^{\mathrm{T}})$ consists of all multiples of the special solution $y = [-(t_2-t_3)/(t_2-t_1), -(t_3 - t_1)/(t_2 - t_1), 1]$. So the condition becomes $y^{\mathrm{T}}[b_1; b_2; b_3] = 0$, or

$$\boxed{-b_1(t_2 - t_3)/(t_2 - t_1), -b_2(t_3 - t_1)/(t_2 - t_1) + b_3 = 0.}$$

Finally, this equation is equivalent to the one displayed above.

4. Problem 1, section 4.4, p.239.

Solution: The pairs are in (a) only independent, in (b) both independent and orthogonal, and in (c) all three. To produce orthonormal vectors, change the second vector in (a) to $\boxed{[0; 1]}$ and in (b) to $[.4; -.3]/\sqrt{.16 + .09} = \boxed{[.8; -.6]}$.

5. Problem 4, section 4.4, p.239.

Solution: Examples are the following: (a) $Q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with $Q\,Q^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$;

(b) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$; and (c) $q_2 = (1, -1, 0)/\sqrt{2}$ and $q_3 = (1, 1, -2)/\sqrt{6}$.

6. Problem 18, section 4.4, p.241.

Solution: The Gram-Schmidt process yields the following:

$A = a = \boxed{(1,\ -1,\ 0,\ 0)}$;

$B = b - p_A = (0,\ 1,\ -1,\ 0) - (1,\ -1,\ 0,\ 0) * (-1)/2 = \boxed{(1/2,\ 1/2,\ -1,\ 0)}$;

$C = c - p_A - p_B = (0,\ 0,\ 1,\ -1) - (1,\ -1,\ 0,\ 0) * (0)/2 - (1/2,\ 1/2,\ -1,\ 0) * (-1)/(1/4 + 1/4 + 1 + 0)$

$= \boxed{(1/3,\ 1/3,\ 1/3,\ -1)}$.

7. Problem 20, section 4.4, p.241.

Solution: (a) True, an example is $Q = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Big/ \sqrt{2}$ with $Q^{-1} = Q^{\mathrm{T}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Big/ \sqrt{2}$.

(b) True as $Q = [\, q_1 \;\; q_2 \,]$ implies $||Q\,x||^2 = (x_1\, q_1^{\mathrm{T}} + x_2\, q_2^{\mathrm{T}}) * (q_1\, x_1 + q_2\, x_2) = {x_1}^2 + {x_2}^2$ since $q_1^{\mathrm{T}} q_1 = 1$, $q_1^{\mathrm{T}} q_2 = 0$ and $q_2^{\mathrm{T}} q_2 = 1$. An example is $Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \end{bmatrix}$ and $x = \begin{bmatrix} \sqrt{3} \\ \sqrt{2} \end{bmatrix}$. Here $Q\,x = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. So $||Q\,x||^2 = 4 + 1 = 5$.

And $||x||^2 = 3 + 2 = 5$.

8. Problem 2, section 8.5, p.451.

Solution: Three integration show that the polynomial 1, x, $x^2 - 1/3$ are orthogonal on the interval $[-1,\, 1]$:

$\int_{-1}^{1} (1)(x)dx = [x^2/2]_{-1}^{1} = 0;$

$\int_{-1}^{1} (1)(x^2 - 1/3)dx = [x^3/3 - x/3]_{-1}^{1} = 2(1/3 - 1/3) = 0;$

$\int_{-1}^{1} (x)(x^2 - 1/3)dx = [x^4/4 - x^2/6]_{-1}^{1} = 0.$

Clearly, any polynomial of degree 2 can be written as a linear combination of 1, x, $x^2 - 1/3$. By inspection, $2x^2 = 2(x^2 - 1/3) + 0(x) = (2/3)(1)$. Those coefficients 2, 0, 2/3 can also be found by integrating $f(x) = 2x^2$ times the three basis functions and dividing by their "length" squared.

9. Problem 4, section 8.5, p.451.

Solution: On $[-1, 1]$, the integrals of any odd function vanishes. So for any c,

$$\int_{-1}^{1} (1)(x^3 - cx)dx = 0 \quad \text{and} \quad \int_{-1}^{1} (1)(x^2 - 1/3)(x^3 - cx)dx = 0.$$

Choose c so that the remaining integral vanishes:

$$\int_{-1}^{1} (x)(x^3 - cx)dx = [x^5/5 - cx^3/3]_{-1}^{1} = 2(1/5 - c/3) = 0.$$

Thus $\boxed{c = 3/5}$.

10. Problem 6, section 8.5, p.451.

Solution: Equations (6) and (8) on p.449 yield

$$2\pi = \pi(4/\pi)^2(1/1^2 + 1/3^2 + 1/5^2 + \cdots) \quad \text{or}$$

$$\boxed{\pi^2 = 8(1/1^2 + 1/3^2 + 1/5^2 + \cdots)}.$$

11. Find the best linear approximation to $y = x^2$ on $[-1, 1]$.

Solution: In problem 2, section 8.5, it was shown that $1, x, x^2 - 1/3$ are orthogonal. By inspection, $x^2 = 1(x^2 - 1/3) + 0(x) + (1/3)(1)$. Hence the orthogonal projection of $y = x^2$ into the span of 1 and x is $\boxed{y = 1/3}$, which is therefore the best linear approximation.