18.06 F10 Problem Set 5 Solutions Due Thursday, 14 Oct.

1. Problem 14, section 8.2, p.430.

Solution: Suppose $A^{\mathrm{T}}CAx = 0$. Set $y := \sqrt{C}Ax$. Then $||y||^2 = x^{\mathrm{T}}(\sqrt{C}A)^{\mathrm{T}}(\sqrt{C}A)x = x^{\mathrm{T}}A^{\mathrm{T}}CAx = 0$. So $y = 0$. We assume no diagonal entry of C is 0. Hence $Ax = 0$. Conversely, if $Ax = 0$, then $A^{\mathrm{T}}CAx = 0$. Thus the vectors x in the nullspace of $A^{\mathrm{T}}CA$ are just the x in the nullspace of A, namely, all multiples of $[1; 1; 1; 1]$.

The equation of $A^{\mathrm{T}}CAx = f$ is solvable if and only if x belongs to the column space of $A^{\mathrm{T}}CA$. By symmetry, the latter is equal to the row space, so is the orthogonal complement of the nullspace. Hence $A^{\mathrm{T}}CAx = f$ is solvable if and only if $[1, 1, 1, 1] * f = 0$, or $f_1 + f_2 + f_3 + f_4 = 0$.

2. Solve the system

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} x = \begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix}.$$

Solution: Performing elimination yields

$$A = \left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 8 \\ 4 & 5 & 6 & 5 \\ 7 & 8 & 9 & 2 \end{array}\right) \xrightarrow[l_{31}=3]{l_{21}=4} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & \boxed{-3} & -6 & -27 \\ 0 & -6 & -12 & -54 \end{array}\right)$$

$$\xrightarrow{l_{32}=2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & -3 & -6 & -27 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

There is just one free variable x_3. So the general solution is

$$x = \begin{pmatrix} -10 \\ 9 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

3. Since A has two pivots $r(A) = 2$.

All ranks of B can be achieved.

- $r(B) = 0$ this can only be the 0 matrix: $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

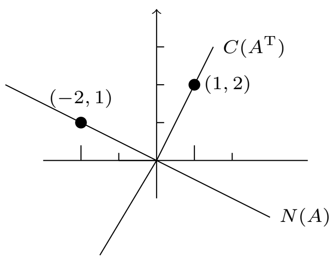
  in this case $AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

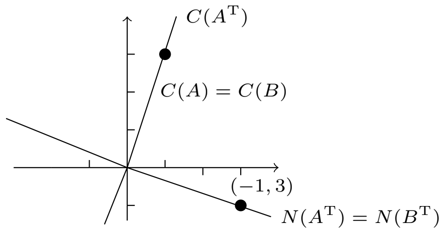
  thus $r(AB) = 0 = \min\{2, 0\} = \min\{r(A), r(B)\}$

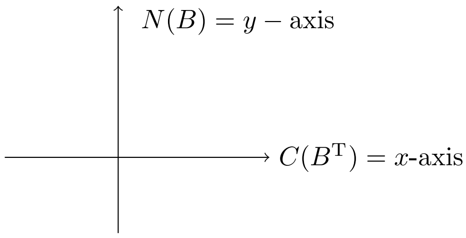
- $r(B) = 1$ We want $r(AB) = \min\{r(A), r(B)\} = \min\{2, 1\} = 1$.
  We know that $r(AB) \leq 1$, thus this holds if AB is not the 0 matrix, which happens whenever B is not of the form
  $$\begin{pmatrix} a & b & c \\ -2a & -2b & -2c \\ a & b & c \end{pmatrix}$$
  thus for example for
  $$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
  We have
  $$AB = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 7 & 0 & 0 \end{pmatrix}$$
  and $r(AB) = 1 = min\{r(A), r(B)\} = 1$
- $r(B) = 2$ For
  $$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
  We have
  $$AB = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 0 \end{pmatrix}$$
  Thus $r(AB) = 2 = \min\{r(A), r(B)\} = \min\{2, 2\} = 2$
- $r(B) = 3$. If B is invertible then $r(AB) = r(A)(= \min\{r(A), 3\})$ always holds.
  Pick for example
  $$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
  Now
  $$AB = A$$
  Thus $r(AB) = r(A) = 2$ as required.

4. Problem 11, sections 4.1, p.203.

Solution: Since $\mathrm{rref}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, we have $\lambda(A) = 1$ and the pivot column of A is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Hence $C(A) = C(B)$ and $N(A^{\mathrm{T}}) = N(B^{\mathrm{T}})$. Thus the correct figures are the following







5. Problem 17, section 4.1, p.204.

Solution: If S is the subspace of IR^3 containing only the zero vector, then $S^{\perp}$ $=\mathrm{IR}^3$. If S is spanned by $(1,1,1)$, then $S^{\perp}$ is the plane with equation $x+y+z=0$, which is the plane spanned by $(1,-1,0)$ and $(1,0,-1)$. If S is spanned by $(1,1,1)$ and $(1,1,-1)$, then $\{(1,-1,0)\}$ is a basic for S^{T}.

6. (a) Entry (i,j) in AA^{T} is the inner product of row i and row j of A.
   (b) Since A is orthogonal, $A^{\mathrm{T}}A = I$, So $A^{\mathrm{T}} = A^{-1}$. Hence $(A^{\mathrm{T}})^{\mathrm{T}}A^{\mathrm{T}} = AA^{\mathrm{T}} = I$. Thus A^{T} is orthogonal.
   (c) If A and B are orthogonal, then $(AB)^{\mathrm{T}}(AB) = B^{\mathrm{T}}A^{\mathrm{T}}AB = B^{\mathrm{T}}B = I$. Thus AB is orthogonal.

7. Problem 5, section 4.2, p.214.

Solution: First, ${a_1}^{\mathrm{T}}a_1 = 1+4+4 = 9$. So $P_1 = \frac{1}{9} * a_1{a_1}^{\mathrm{T}} = \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$.
Second, ${a_2}^{\mathrm{T}}a_2 = 4+4+1 = 9$. So $P_2 = \frac{1}{9}a_2{a_2}^{\mathrm{T}} = \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$. Finally, $P_1P_2 = 0$ because $a_1 \perp a_2$ as ${a_1}^{\mathrm{T}}a_2 = -2+4-2 = 0$.

8. <u>Problem 14, section 4.2, p.215.</u>

<u>Solution</u>: The projection of b is itself, because b lies in the column space of A. No, $P \neq I$ if $C(A) \neq \mathrm{IR}^{rn}$. When $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{pmatrix}$, then $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = \frac{1}{21}\begin{pmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{pmatrix}$ and $b = Pb = p = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$.

9. <u>Problem 17, section 4.2, p.215.</u>

<u>Solution</u>: If $P^2 = P$, then $(I-P)^2 = (I-P)(I-P) = I-PI-IP+P^2 = I-P$. When P projects onto the column space of A, then $I \cdot P$ projects onto its orthogonal complement, which is the left nullspace of A.