Solutions to 18.06 Problem Set 4

1. The matrix is already given in LU form:

$$\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}\begin{pmatrix}0&1&2&3&4\\0&0&0&1&2\\0&0&0&0&0\end{pmatrix}$$

The row space and null space for A and for U are the same, so we do it for U. From U, we can read off the row space as the span of $(0,1,2,3,4)$ and $(0,0,0,1,2)$.

For the null space of U, we identify the free variables as x_1, x_3, x_5. So first set $x_1 = 1$ and $x_3 = x_5 = 0$. The solution is $(1,0,0,0,0)^T$. Now set $x_1 = x_5 = 0$ and $x_3 = 1$. The solution is $(0,-2,1,0,0)^T$. Finally, set $x_1 = x_3 = 0$ and $x_5 = 1$. The solution is $(0,2,0,-2,1)^T$. So the span of these three vectors is the null space of both U and A.

For the column space of A, we first take a spanning set for the column space of U and then apply L to these vectors. The column space of U is the span of the columns with pivots, so the column space of U is the span of $\{(1,0,0)^T, (3,1,0)^T\}$. Applying L gives that the column space of A is $\{(1,1,0)^T, (3,4,1)^T\}$.

The kernel of A^T is spanned by the last row of $E = L^{-1}$ since the last row of U is all 0's. Then $L^{-1} = \begin{pmatrix}1&0&0\\-1&1&0\\1&-1&1\end{pmatrix}$, so the answer is $(1,-1,1)$.

2. $A = \begin{pmatrix}1&1&1\\2&1&0\end{pmatrix}$ and $B = \begin{pmatrix}-1&2&-1\end{pmatrix}$ (idea for B: we need that two times column 1 plus column 2 is 0 since $(2,1,0)$ is in null space. This is easy to find; then pick the third column so that the whole sum is 0 since $(1,1,1)$ is in null space)

3. $A = \begin{pmatrix}1&2\\2&2\\4&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}$. The vectors are $u = \begin{pmatrix}1\\2\\4\end{pmatrix}$, $v = (1,0)$, $w = \begin{pmatrix}2\\2\\1\end{pmatrix}$, $z = (1,1)$. (explanation: the column space of the product is contained in the column space of the first matrix and the row space of the product is contained in the row space of the second matrix. The spaces are actually equal because both matrices have rank 2 which is the dimension of the row and column spaces).

4. The matrix of interest is $\begin{pmatrix}-1&1&0\\-1&0&1\\0&-1&1\end{pmatrix}$. Row reducing gives the matrix $\begin{pmatrix}-1&1&0\\0&-1&1\\0&0&0\end{pmatrix}$. This gives the tree $1 \xrightarrow{1} 2 \xrightarrow{2} 3$.

5. The incidence matrix is $A = \begin{pmatrix}-1&1&0&0\\-1&0&1&0\\0&-1&1&0\\0&-1&0&1\\0&0&-1&1\end{pmatrix}$. As with every incidence matrix, the all 1's vector $(1,1,1,1)$ gives an element in the null space.

For solutions to $A^T y = 0$, we can go around loops in the graph. One loop goes through edge 1, through edge 3, then against edge 2, so we get a solution $(1,-1,1,0,0)^T$. Another loop goes through edge 3, through edge 5, and against edge 4, so we get a solution $(0,0,1,-1,1)^T$.