18.06 Problem Set 3 Solutions

1. Find the LU and the reduced row echelon matrix form for

$$A = \begin{pmatrix} 2 & 4 & 6 & 8 & 4 \\ 4 & 11 & 15 & 24 & 14 \\ 2 & 10 & 12 & 28 & 24 \end{pmatrix}.$$

Compute the column space $C(A)$ and the null space $N(A)$ for A. Give all solutions for the system

$$Ax = \begin{pmatrix} 8 \\ 25 \\ 34 \end{pmatrix}.$$

Solution The steps going to the row-echelon matrix E (which goes through U in the middle) is as follows. To save work for later we'll append the solution vector $\begin{pmatrix} 8 \\ 25 \\ 34 \end{pmatrix}$ as we reduce:

$$\begin{pmatrix} 2 & 4 & 6 & 8 & 4 & 8 \\ 4 & 11 & 15 & 24 & 14 & 25 \\ 2 & 10 & 12 & 28 & 24 & 34 \end{pmatrix} \to \begin{pmatrix} 2 & 4 & 6 & 8 & 4 & 8 \\ 0 & 3 & 3 & 8 & 6 & 9 \\ 0 & 6 & 6 & 20 & 20 & 26 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & 4 & 6 & 8 & 4 & 8 \\ 0 & 3 & 3 & 8 & 6 & 9 \\ 0 & 0 & 0 & 4 & 8 & 8 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 4 & 2 & 4 \\ 0 & 1 & 1 & 8/3 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & -4/3 & -2 & -2 \\ 0 & 1 & 1 & 8/3 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 0 & 2/3 & 2/3 \\ 0 & 1 & 1 & 0 & -10/3 & -7/3 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{pmatrix}.$$

Thus, we have $R = \begin{pmatrix} 1 & 0 & 1 & 0 & 2/3 \\ 0 & 1 & 1 & 0 & -10/3 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$. Tracking the linear combinations used to get $U = \begin{pmatrix} 2 & 4 & 6 & 8 & 4 \\ 0 & 3 & 3 & 8 & 6 \\ 0 & 0 & 0 & 4 & 8 \end{pmatrix}$, which appeared in the second row, we get that $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$.

The column space is spanned by the pivot columns of A, which are columns $1, 2, 4$. Thus, it is the set $x_1 \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 11 \\ 10 \end{pmatrix} + x_4 \begin{pmatrix} 8 \\ 24 \\ 28 \end{pmatrix}$.

We can get the null space by setting the free variables x_3 and x_5 to what we want and then solving the equation $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$. A shortcut (though equivalent) is that we can replace A with R in the above equation - we can even read them directly from the free columns. $x_3 = 1, x_5 = 0$ gives the vector $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $x_3 = 0, x_5 = 1$ gives the vector $\begin{pmatrix} -2/3 \\ 10/3 \\ 0 \\ -2 \\ 1 \end{pmatrix}$, which we can then scale to $\begin{pmatrix} -2 \\ 10 \\ 0 \\ -6 \\ 3 \end{pmatrix}$.

So we can write the null space as $a \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -2 \\ 10 \\ 0 \\ -6 \\ 3 \end{pmatrix}$.

Now, to get all solutions to the given equation, we can start with any particular solution and add anything we want in the null space. To get a particular solution, we can just set the "free" variables to anything we want in x and solve $Ax = 0$ (or use back-substitution and LU). However, we can also do what we did here, which is just to transform the b along with the A. Here, we can set the free variables to 0 and solve $E \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -7/3 \\ 2 \end{pmatrix}$ to get a particular solution $(2/3, -7/3, 0, 2, 0)$. However, you could have also, say, have set $x_2 = x_5 = 1$, because that gives us

$$\begin{pmatrix} 2 & 4 & 6 & 8 & 4 \\ 0 & 3 & 3 & 8 & 6 \\ 0 & 6 & 6 & 20 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ 1 \\ x_3 \\ x_4 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 25 \\ 34 \end{pmatrix},$$

which we can rewrite as:

$$\begin{pmatrix} 2 & 6 & 8 \\ 0 & 3 & 8 \\ 0 & 6 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so another particular solution is $(0, 1, 0, 0, 1)$.

We can pick any particular solution and add the nullspace, so one version would be:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + a \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -2 \\ 10 \\ 0 \\ -6 \\ 3 \end{pmatrix}$$

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2. Do problem 9 from section 3.2.

   Solution

   (a) False. Any matrix with fewer than full number of pivots will. Example: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

   (b) True. Since it is invertible, we will get the full number of pivots. Equivalently, the nullspace has dimension 0, so we have 0 free variables.

(c) True. The number of pivot variables is the dimension of the nullpace, which is at most the number of columns (recall that the nullspace dimension + the column space dimension = number of columns).

(d) True. You can see this because in the reduced echelon matrix the pivot columns are all 0 except for a single 1, and there are only up to m vectors of this type!

The purpose of the last two is this: the number of pivots is the rank, which cannot exceed either m or n. The rank is a nice concept since it doesn't treat rows and columns differently, so there's something very aesthetically nice about it compared to some of the other concepts we talked about.

3. Do problem 25 from section 3.2.

   Solution We obviously need a 3×4 matrix. Think about what the nullspace means. Each row a, b, c, d must send $1, 1, 1, 1$ to 0, so $a + b + c + d$=0. Getting the column space to contain $(1, 1, 1)$ is easy: we can just make the column that (or its negative). So one example that works is:

$$P = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

   This is actually a fairly loaded (and annoying) question in this sense: if we have a 3×4 matrix and the nullspace is exactly the multiples of $(1, 1, 1, 1)$, then the column space *must* contain $(1, 1, 1)$. Do you see why?

4. Do problem 8 from section 3.3.

   Solution Rank = 1 means that all the rows are multiples of each other (and the same is true for the columns!). This is really all it takes to get the matrices:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{pmatrix}, B = \begin{pmatrix} 3 & 9 & -9/2 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{pmatrix}, M = \begin{pmatrix} a & b \\ c & bc/a \end{pmatrix}.$$

5. Do problem 23 from section 3.3.

   Solution For A, we can kill the second row immediately. Now, there are two cases:

   (a) $c = 1$, in which case when we subtract row 1 from row 3 row 3 goes to 0, so we get the row-reduced form: $\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. This has only one pivot (the first column) and 3 free variables x_2, x_3, x_4, so we can do the usual and substitute 1 for each x_i and send the other to 0 to get the nullspace matrix $\begin{pmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   (b) c is not 1, in which case the subtraction gives us $(0, c-1, 0, 0)$ which we can then scale to $(0, 1, 0, 0)$, and we get $\begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Now we have two pivots (the first two) and 2 free variables x_3, x_4. Substituting 1 for them in turn gives the nullspace matrix $\begin{pmatrix} -2 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

For B, this is a little more annoying. There are now 3 cases:

(a) $c = 1$, in which case after reduction we get we can substract the second row from the first to get $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. This has a single pivot in the second column and one free variable, with the nullspace matrix $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(b) $c = 2$, in which case we can multiply the first row by -1 to get $\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$. the first column is a pivot, and we get the nullspace matrix $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(c) Otherwise, we can always multiply the first row by $1/(1-c)$, the second row by $1/(2-c)$, and then make the upper right entry 0. This gives us the identity matrix with 2 pivots, which has $2 - 2 = 0$ null space.

6. Do problem 3 from section 3.4.

Solution We append the solution column to the right and row reduce, to get

$$\begin{pmatrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Restricting to the left three columns (careful! The rightmost column is *not* in the upper-triangular form of A) to get $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, we can see that the 2nd column is a free variable. We can set x_2 to 0 to find a particular solution $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$. We get the solution $(-2, 0, 1)$.

Now, the general solution will be this particular solution plus the nullspace. The nullspace has dimension 1 (because there's just 1 free variable), so we set $x_2 = 1$ and solve $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ 1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. This gives the solution $(-3, 1, 0)$.

Thus, the general solution is $(-2, 0, 1) + a(-3, 1, 0)$.

7. Do problem 32 from section 3.4.

Solution

$$\begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & -2 & 4 \\ 0 & -2 & 4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = U$$

Note we have 2 pivots and one free variable, so the solution will have a nullspace of dimension 1.

Tracking the multiples we used to eliminate the rows, we get get $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$.

Now let's find some particular solutions for the two b given. This involves solving $Lx = b$ and doing back-substitution into U, or we could have also appended the b as a column while doing the row reductions. Let's first find the null space, which should be dimension 1. Substituting 1 for the third variable and solving $Ux = 0$ gives $(-7, 2, 1)$, so the nullspace is a line along that (of course we could have also solved $Ax = 0$; the nullspaces are the same).

For $b = (1, 3, 6, 5)$: b is exactly the third column of A and must then be sent to $(1, 2, 0, 0)$ (if we had appended it to the right of A and done upper-triangulation on the augmented matrix). Since the third column is a free variable we can assume it is 0, and a particular solution is $(7, -2, 0)$. So the general solution is $(7, -2, 0) + a(-7, 2, 1)$.

For $b = (1, 0, 0, 0)$, I'll use the back-substitution method. Since we want $Ax = LUx = b$, we first solve $Ly = b$. We get $y_1 = 1, y_2 = -1, y_3 = 0, y_4 = 1$ in turn. So now we can solve $Ux = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$. But this is impossible, so there is no solution.

8. Do problem 10 from section 3.5.

Solution $(2, -1, 0, 0)$ and $(1, 1, 1)$ for example, is an independent pair (there are lots of these).

Directly finding the independent vectors by trial and error is not too hard, but here's a more conceptual way given what we've learned:

Consider the matrix equation $[1 \;\; 2 \;\; -3 \;\; -1]b = 0$. The possible $b = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ is exactly the plane (in other words, our plane is the nullspace of the matrix $[12 - 3 - 1]$). This matrix is already in row-echelon form, with y, z, t being the free variables. So setting each one to 1 and the others in 0 in turn should give us three independent vectors of the nullspace, which it does! They give $[-2, 1, 0, 0], [3, 0, 1, 0], [1, 0, 0, 1]$ respectively, which are linearly independent.

Note we can't find four independent vectors because the nullspace only has dimension 3 (having 3 free variables). Alternatively, you can just know that the plane has dimension $4 - 1 = 3$ and we can't have 4 independent vectors in 3-dimensions.

9. Do problem 26 from section 3.5.

Solution Let E_{ij} be the matrix with a single 1 in the i-th row and j-th column. For example, $E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(a) A basis would be E_{11}, E_{22}, E_{33}, so the dimension is 3.

(b) A 6-dimensional basis: $E_{11}, E_{22}, E_{33}, E_{12} + E_{21}, E_{13} + E_{31}, E_{23} + E_{32}$.

(c) A 3-dimensional basis: $E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32}$. Notice we get fewer because the diagonal terms are forced to be zero.