18.06 Problem Set 1 Solutions

1. Find a solution for x,y,z to the system of equations

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3\,e + \pi + 2\sqrt{2} \\ 6\,e + 4\,\pi + 5\sqrt{2} \\ 10\,e + 7\,\pi + 8\sqrt{2} \end{pmatrix}$$

Solution $x = \pi$, $y = \sqrt{2}$ and $z = e$ is a solution.

2. Do problem 11 from section 2.1.

Solution by rows:

$$\begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} (2\ 3) \cdot (4\ 2) \\ (5\ 1) \cdot (4\ 2) \end{pmatrix} = \begin{pmatrix} 14 \\ 22 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} (3\ 6) \cdot (2\ -1) \\ (6\ 12) \cdot (2\ -1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} (1\ 2\ 4) \cdot (3\ 1\ 1) \\ (2\ 0\ 1) \cdot (3\ 1\ 1) \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

Solution by columns:

$$\begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ 22 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 6 \end{pmatrix} - 1\begin{pmatrix} 6 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

3. Do problem 26 from section 2.1.

Solution

The matrix form for the system of equations:

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

On the row picture the two lines are the lines defined by the equations $x - 2y = 0$ and $x + y = 6$, and their intersection is the point $(2\ 4)$ the solution for the system.

The column picture pictures the column vectors: $(1\ 1)$ and $(-2\ 1)$. The paralelogram show how the solution vector $(0\ 6)$ can be written as the linear combination of the column vectors.

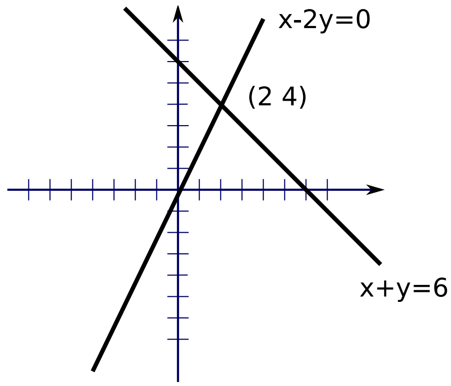

Figure 1: row picture

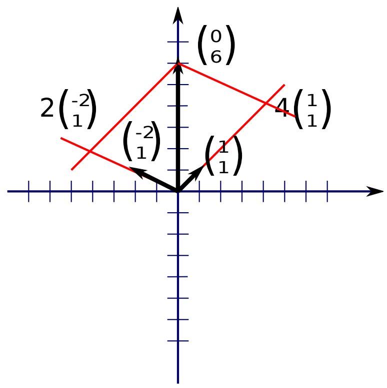

Figure 2: column picture

4. Do problem 7 from section 2.2.

   Solution

   The matrix form for the system of equations:

$$\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

   The first pivot is a, thus for $a = 0$ the elimination brakes down temporarily.

   If $a = 0$ exchange the two equations:

$$\begin{pmatrix} 4 & 6 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

   The system is in upper triangular form, with nonzero entries in the diagonal, thus it has a unique solution.

   If $a \neq 0$, then subtract $\frac{4}{a} \cdot$ "first equation" from the "second equation" to get:

$$\begin{pmatrix} a & 3 \\ 0 & 6 - 3\frac{4}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + 3\frac{4}{a} \end{pmatrix}$$

The second pivot is $6 - 3\frac{4}{a}$, and if $6 - 3\frac{4}{a} \neq 0$ (that is if $a \neq 2$), then the matrix is upper triangular with nonzero entries in the diagonal, thus the system has a unique solution.

If $a = 2$, then the system looks like:

$$\begin{pmatrix} 8 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}$$

The elimination permanently failed, the system has no solutions.

Summarizing, the elimination temporary breaks down if $a = 0$, and permanently if $a = 2$.

5. Do problem 31 from section 2.2.

Solution The elimination process uses only the rows over the j'th row, thus row j is the combination of row 1, row 2, ... and row j.

This is also true for the coordinates of the solution vector $\mathbf{b} = (b_1\ b_2\ \ldots\ b_n)$, thus the number b_j is a combination of $b_1, b_2, \ldots, b_j$. If $\mathbf{b} = \mathbf{0}$ this means, that after the elimination the j'th coordinate of the new solution vector is a combination $b_1, b_2, \ldots, b_j$ which are all 0s. Thus the new solution vector is $\mathbf{0}$ too.

However if $\mathbf{b} \neq \mathbf{u}$ then the coordinates of the new solution vector are going to be combinations of $b_1, b_2, \ldots, b_n$, but nort neccesarry the same. For example for

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and $\mathbf{b} = (1\ 2)$ the system is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and after elimination:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so the new solution vector $(1\ 1) \neq \mathbf{b}$.

If A is lower triangular, what is the upper triangular U?

As A is lower triangular during the elimination process the diagonal elemnents won't change. The pivots are going to be the diagonal elements, thus if none of the diagonal entries are 0 U is the diagonal matrix with the same diagonal as A.

6. Do problem 21 from section 2.3.

Solution

One can directly check, by writting down the exact matrices in the 3 dimensional case:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } F = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then EF is the matrix formed from F by adding its' 1st row two its' 2nd row (or directly):

$$EF = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly FE is the matrix formed from E by adding its' 2nd row two its' 1st row (or directly):

$$FE = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So $EF \neq FE$.

7. Do problem 23 from section 2.4.

   Solution For

   $$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   $A^2 = 0$. And for

   $$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

   $$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   and $A^3 = 0$.

8. Do problem 32 from section 2.4.

   Solution

   The columns of AX are exactly $Ax_1 = (1\ 0\ 0)$, $Ax_2 = (0\ 1\ 0)$ and $Ax_3 = (0\ 0\ 1)$, thus

   $$AX = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   .

9. Do problem 6 from section 2.5.

   Solution

   (a) $AB = AC$ multiple the equation from the left by A^{-1} (as matrix multiplication is not commutative you do need to specify the side you multiple from). We get

   $$\begin{aligned} AB &= AC \\ A^{-1}(AB) &= A^{-1}(AC) \\ (A^{-1}A)B &= (A^{-1}A)C \\ B &= C \end{aligned}$$

   We are done.

   (b) For

   $$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

   Let

   $$B = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

Then even though $B \neq C$

$$AB = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = AC$$

(c) In the language of your choice, write a function “rowop” that takes a matrix A and replaces the second row with the original second row minus 10 times the first row.

(*Hint:*In Matlab you can create a file with the name "rowop.m" with header function $B = \text{rowop}(A)$ Useful commands are $A(2,:) * 10$ and $A(2,:) =$)

Solution see separate sheet