18.06 Professor Edelman Final Exam December 15, 2010

Your PRINTED name is: ____________________

Please circle your recitation:

R01	T 9	2-132	S. Kleiman	2-278	3-4996	`kleiman`
R02	T 10	2-132	S. Kleiman	2-278	3-4996	`kleiman`
R03	T 11	2-132	S. Sam	2-487	3-7826	`ssam`
R04	T 12	2-132	Y. Zhang	2-487	3-7826	`yanzhang`
R05	T 1	2-132	V. Vertesi	2-233	3-2689	`18.06`
R06	T 2	2-131	V. Vertesi	2-233	3-2689	`18.06`

Grading

1

2

3

4

5

6

7

8

9

1 (16 pts.)

a. (8 pts) Give bases for each of the four fundamental subspaces of $A = \begin{bmatrix} 1 & 0 & \pi & e \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clearly the first two columns are independant and generate the column space.

The left null space is the orthogonal of the column space. A basis is given by the vector :

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

A possible basis for the nullspace is :

$$\begin{bmatrix} \pi \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} e \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

The row space has dimension 2. The two first rows clearly generate it, therefore they form a basis.

b. (8 pts) Give bases for each of the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(Each of the three matrices in the above product has orthogonal columns.)

The matrices on both side are non singular. Therefore the rank of the product is 3.

To compute the column space, we can restrict to the product of the first two matrices. The third is invertible and will not change the column space. Clearly the column space of the product of the first two matrices is spanned by the first three columns of the leftmost matrix. Since these vectors are independant they form a basis.

To compute the row space we can restrict to the last two matrices of the product for the same reason as above. We find that the row space has as basis the first three rows of the rightmost matrix.

Since the nullspace is the orthogonal of the row space, and we know that the rows of the rightmost matrix are orthogonal, the last row of the right most matrix is a basis for the nullspace.

Similarly, the last column of the leftmost matrix is a basis for the left nullspace.

Remark. Another way to see it is to notice that this is almost the SVD of A (we just need to normalize the columns of the leftmost and rightmost matrix). Our answer is exactly the usual way to find a basis of the four fundamental subspaces when we have found the SVD.

This page intentionally blank.

2 (14 pts.)

Let P_1 be the projection matrix onto the line through $(1, 1, 0)$ and P_2 is the projection onto the line through $(0, 1, 1)$.

(a) (4 pts) What are the eigenvalues of P_1?

The eigenvalue of a projection matrix are 1 and 0. The question remain of the multiplicity of each eigenvalue. The multiplicity of 0 is the dimension of the nullspace which is $3-$ the rank. Here P_1 is a projection on a line, therefore the columnspace is made of vector on that line and has dimension 1. Therefore 0 has multiplicity 2 and 1 has multiplicity 1.

(a2)(bonus) (This question is from an earlier version of the exam.)

Find an eigenvalue and an eigenvector of $P_1 + P_2$.

The problem asks only for one eigenvalue and one eigenvector, but since you're taking this exam for practice, you may as well find all three.

For simplicity of notation, let $\vec{a}_1 := (1, 1, 0)$ and $\vec{a}_2 := (0, 1, 1)$. Note that the vectors $\vec{a}_1$ and $\vec{a}_2$ have the same length, $\sqrt{2}$. Since $\vec{a}_1 \cdot \vec{a}_2 = 1 = 2\cos(60°)$, we know that the angle between $\vec{a}_1$ and $\vec{a}_2$ is $60°$. It is possible to see the following facts geometrically:

- First, $\vec{v}_1 := \vec{a}_1 \times \vec{a}_2 = (1, -1, 1)$ is an eigenvector of $P_1 + P_2$, with eigenvalue $\lambda_1 := 0$. Indeed, since v_1 is perpendicular to both $(1, 1, 0)$ and $(0, 1, 1)$, we know that v_1 lies in the nullspace of both P_1 and P_2, and hence in the nullspace of $P_1 + P_2$ as well.
- Second, $\vec{v}_2 := \vec{a}_1 + \vec{a}_2 = (1, 2, 1)$ is an eigenvector of $P_1 + P_2$, with eigenvalue $\lambda_2 := 2\cos^2(60°/2) = 1 + \cos(60°) = 3/2$.
- Third, $\vec{v}_3 := \vec{a}_1 - \vec{a}_2 = (1, 0, -1)$ is an eigenvector of $P_1 + P_2$, with eigenvalue $\lambda_3 := 2\sin^2(60°/2) = 1 - \cos(60°) = 1/2$.

The second and third facts can be derived with a bit of trigonometry, but if you don't want to get into that, you can just do the usual linear-algebra calculation. Note that

$$P_1 = \frac{1}{1^2+1^2+0^2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and similarly

$$P_2 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

so that

$$P_1 + P_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

You may solve for the roots $\lambda_1, \lambda_2, \lambda_3$ of $\det(P_1 + P_2 - \lambda I) = 0$ and find the nullspaces of $P_1 + P_2 - \lambda_i I$ (for $i = 1, 2, 3$) as usual, and get the results that we described above.

(b) (10 pts) Compute $P = P_2 P_1$. (Careful, the answer is not 0)

Let's just do it directly:

$$P = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

This page intentionally blank.

3 (10 pts.)

The nullspace of the matrix A is exactly the multiples of $(1, 1, 1, 1, 1)$.

(a) (2 pts.) How many columns are in A?

Five columns. Otherwise it doesn't make sense to multiply A with the given vector.

(b) (3 pts.) What is the rank of A?

We know that the sum of the rank of A and the dimension of the nullspace is the number of columns. Since the nullspace has dimension 1, the rank must be 4.

(c) (5 pts.) Construct a 5×5 matrix A with exactly this nullspace.

We need to construct a matrix whose sum of columns is 0 and with rank 4.

The following works :

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This page intentionally blank.

4 (15 pts.)

Find the solution to

$$\frac{du}{dt} = -\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} u$$

starting with $u(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

(Note the minus sign.)

The general formula for the solution to this differential equation is entirely analogous to the formula you learned in 18.01:

$$\vec{u}(t) = e^{-t\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}}\vec{u}(0).$$

Of course, you should actually rewrite this in simplest form.

To facilitate taking the matrix exponential, let's diagonalize

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

The eigenvalues are easily computed to be 0 (because the matrix is obviously singular) and 3 (because the trace is 3), and the corresponding eigenvectors are $(2,-1)$ and $(1,1)$, so

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1}.$$

It follows that

$$e^{-t\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} e^{-t\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{-0t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1}.$$

Now

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix},$$

so

$$
\begin{aligned}
\vec{u}(t) &= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{-0t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\vec{u}(0) \\
&= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ e^{-3t} \end{bmatrix} \\
&= \begin{bmatrix} e^{-3t} + 2 \\ e^{-3t} - 1 \end{bmatrix}.
\end{aligned}
$$

This page intentionally blank.

5 (10 pts.)

The 3×3 matrix A satisfies $\det(tI - A) = (t-2)^3$.

(a) (2pts) What is the determinant of A?

If we take $t = 0$, we find $\det(-A) = -8$. Since A is 3×3, $\det(-A) = -\det(A)$. Therefore $\det(A) = 8$.

(b) (8pts) Describe all possible Jordan normal forms for A.

The eigenvalues of A are 2 with multiplicity 3. The possible Jordan form of A are :

$$\begin{bmatrix} 2 & 1 \text{ or } 0 & 0 \\ 0 & 2 & 1 \text{ or } 0 \\ 0 & 0 & 2 \end{bmatrix}$$

There are 4 possibilities.

This page intentionally blank.

6 (7 pts.)

The matrix $A = \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix}$

(a) (2 pts) What are the eigenvalues of A?

This is a triangular matrix, therefore the eigenvalues are the diagonal entries 1 and 1.

(b) (5 pts) Suppose σ_1 and σ_2 are the two singular values of A. What is $\sigma_1^2 + \sigma_2^2$?

σ_1^2 and σ_2^2 are the eigenvalues of $A^T A$. The question is asking for the trace of $A^T A$.

$$A^T A = \begin{bmatrix} 1 + C^2 & ? \\ ? & 1 \end{bmatrix}$$

(The question marks are entries that we don't care about)

The trace is $2 + C^2$.

This page intentionally blank.

7 (8 pts.)

For each transformation below, say whether it is linear or nonlinear, and briefly explain why.

(a) (2 pts) $T(v) = v/\|v\|$

Not linear. For a positive real number c, $T(cv) = T(v)$ and if T was linear we would have $T(cv) = cT(v)$.

(b) (2 pts) $T(v) = v_1 + v_2 + v_3$

Linear. We have $T(0) = 0$, $T(v + w) = T(v) + T(w)$ and $T(cv) = cT(v)$.

(c) (2 pts) $T(v) =$ smallest component of v

Not linear. Let's say that v is 2-dimensional. Take $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $T(-v) = -1 \neq -T(v) = 0$.

(d) (2 pts) $T(v) = 0$

Linear. Clearly satisfies all the requirement.

This page intentionally blank.

8 (10 pts.)

V is the vector space of (at most) quadratic polynomials with basis $v_1 = 1$,$v_2 = (x-1)$,$v_3 = (x-1)^2$. W is the same vector space, but we will use the basis $w_1, w_2, w_3 = 1$,x,x^2.

(a) (5 pts) Suppose $T(p(x)) = p(x+1)$. What is the 3×3 matrix for T from V to W in the indicated bases?

We have to compute $T(v_1)$, $T(v_2)$, $T(v_3)$ and write them in the basis w_1, w_2, w_3. We have $T(v_1) = 1 = w_1$, $T(v_2) = (x+1-1) = x = w_2$ and $T(v_3) = (x-1+1)^2 = w_3$. Therefore the matrix of T in these basis is the identity matrix :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that since the two basis are different, this does not imply that T is the identity.

(b) (5 pts) Suppose $T(p(x)) = p(x)$. What is the 3×3 matrix for T from V to W in the indicated bases?

We do the same thing, $T(v_1) = w_1$, $T(v_2) = x - 1 = w_2 - w_1$, $T(v_3) = x^2 - 2x + 1 = w_3 - 2w_2 + w_1$. Therefore the matrix of T is :

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

This page intentionally blank.

9 (10 pts.)

In all of the following we are looking for a real 2×2 matrix or a simple and clear reason that one can not exist.

Please remember we are asking for a real 2×2 matrix.

(a) (2 pts) A with determinant -1 and singular values 1 and 1.

Take $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Clearly $|A| = -1$ and $A^T A = I$ therefore, the singular values are 1 and 1.

(b) (2 pts) A with eigenvalues 1 and 1 and singular values 1 and 0.

This is impossible. If the eigenvalues are 1 and 1, the matrix is non-singular, therefore it has two non-zero singular values (in general the number of non-zero singular values is the rank of the matrix).

(c) (2 pts) A with eigenvalues 0 and 0 and singular values 0 and 1

Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(d) (2 pts) A with rank $r = 1$ and determinant 1

Impossible. If the rank is 1, the matrix is singular and has determinant 0.

(e) (2 pts) A with complex eigenvalues and determinant 1

The determinant is the product of the eigenvalues, hence we need to find complex numbers whose product is 1. One possibility is i and $-i$. We need to find a real matrix with i and $-i$ as eigenvalues. The following matrix works :

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Bonus problem (From an earlier version of the exam)

A basis for the nullspace of the matrix A consists of the three vectors :

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the column space of the matrix A consists of the two vectors :

$$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

(a) (2 pts.) How many rows and how many columns are in A?

If A is an $m \times n$ matrix, then the nullspace $N(A)$ is a subspace of $\mathbb{R}^n$, while the column space $C(A)$ is a subspace of $\mathbb{R}^m$. In our case, this tells us $n = 5$ and $m = 3$. So A is a 3×5 matrix; A has 3 rows and 5 columns.

(b) (6 pts.) Provide a basis for $N(A^T)$ and $C(A^T)$.

The left-nullspace of A, i.e., $N(A^T)$, is the orthogonal complement of $C(A)$ in $\mathbb{R}^3$. Let's take the cross product of the two vectors in the basis for $C(A)$:

$$(1, -1, 1) \times (2, -1, 0) = (1, 2, 1).$$

The vector $(1, 2, 1)$ is perpendicular to the plane $C(A)$, so it's a basis for the line that is orthogonal to this plane, i.e., the line $N(A^T)$. But suppose you didn't think of using the cross product; what could you have done instead? Well, $N(A^T)$ is the space of solutions (x_1, x_2, x_3) to the equation

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let's put the 2×3 matrix in reduced row-echelon form:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}.$$

Now x_3 is a free variable; set it to 1 and use back substitution to find x_1 and x_2. You get exactly $(x_1, x_2, x_3) = (1, 2, 1)$ as above; this special solution forms a basis for $N(A^T)$.

Similarly, the row space of A, i.e., $C(A^T)$, is the orthogonal complement of $N(A)$, so it consists of solutions $(x_1, \ldots, x_5)$ to the equation

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -2 & 0 & 1 & 0 \\ 6 & 5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

One could solve this by Gaussian elimination, but it's probably easiest to observe that, *if you reversed the order of the columns*, the 3×5 matrix would already be in reduced row-echelon form, so the special solutions can be found by setting (x_1, x_2) to $(1, 0)$ or $(0, 1)$, and solving for (x_3, x_4, x_5). In this way, we get the special solutions $(1, 0, -1, -3, -6)$ and $(0, 1, 2, 2, -5)$; these two vectors form a basis for $C(A^T)$.

(c) (6 pts.) Write down an example of such a matrix A.

Thanks to our answer to (b), we know that A has the same column space as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -2 \end{bmatrix}$$

(you could also just say

$$\begin{bmatrix} 1 & 2 \\ -1 & -1 \\ 1 & 0 \end{bmatrix}$$

here, but our method makes the subsequent computations a bit easier). Also from our answer to (b), we know that A has the same row space as

$$\begin{bmatrix} 1 & 0 & -1 & -3 & -6 \\ 0 & 1 & 2 & 2 & -5 \end{bmatrix}.$$

So A is any matrix of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -2 \end{bmatrix} B \begin{bmatrix} 1 & 0 & -1 & -3 & -6 \\ 0 & 1 & 2 & 2 & -5 \end{bmatrix},$$

where B is an invertible 2×2 matrix (if you don't see why, please ask). For example, if we take B to be the identity, we get

$$A = \begin{bmatrix} 1 & 0 & -1 & -3 & -6 \\ 0 & 1 & 2 & 2 & -5 \\ -1 & -2 & -3 & -1 & 16 \end{bmatrix}.$$