18.06 Fall 2010 Exam 3 Solutions

(1) (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) No example exists. There are many ways to see this.

First way: Negative definite means that all upper left submatrices have negative determinant. In particular, the $(1, 1)$ entry needs to be negative, but this violates the definition of Markov.

Second way: Since the trace is the sum of the eigenvalues, all negative eigenvalues implies that the trace is negative. But the diagonal entries are nonnegative by the Markov property, which is another violation.

Third way: A Markov matrix always has 1 as an eigenvalue. All eigenvalues of negative definite are negative.

(c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Really, any matrix of the form $\begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$ where $1/2 > a \geq 0$ would do: then the trace would be < 1 and since the trace is the sum of the eigenvalues and we know that one of the eigenvalues is 1, this means the other one has to be negative.

(d) Take a Jordan block: $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$.

(e) The answer we were looking for: all symmetric real matrices are diagonalizable. So if A is symmetric and has 3 as both of its eigenvalues, then its Jordan normal form is $3I$, and A is similar to $3I$. But the only matrix that is similar to $3I$ is $3I$ itself, which means A would have to be $3I$. Hence no example exists.

Another approach people tried but usually didn't fully justify: write the matrix as $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. The trace is the sum of the eigenvalues and hence 6, and the determinant is the product of the eigenvalues and hence 9. So $a + c = 6$ and $ac - b^2 = 9$. Since $b^2 \geq 0$, we get $ac \geq 9$. The way to maximize ac subject to $a + c = 6$ is to set $a = c = 3$. This can be justified with elementary calculus or citing the arithmetic-mean/geometric-mean inequality (AM-GM) but not many people really explained clearly why this was true. Anyway, this means that $a = c = 3$ and $b = 0$, but then $A = 3I$ again.

(f) $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

(2) (a) A is clearly rank 1, so a reduced SVD like we saw in class is the easiest:

$$A = U\Sigma V^T = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} (1) \begin{pmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{pmatrix}^T$$

(b) **Answer 1**: Notice that $A^2 = -A$, so $e^{At} = I + A(t - t^2/2 + t^3/3! - t^4/4! + \cdots)$. Thus $f(t) = 1 - e^{-t}$.

Answer 2: The eigenvalues of the symmetric rank 1 matrix A are $-1, 0, 0, 0$. In matrix language $\Lambda = \begin{pmatrix} -1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$ and $e^{\Lambda t} - I = \begin{pmatrix} e^{-t} - 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} = (1 - e^{-t})\Lambda$.

If $A = Q\Lambda Q^T$, then $e^{At} - I = Q(e^{\Lambda t} - I)Q^T = (1 - e^{-t})Q\Lambda Q^T = (1 - e^{-t})A$ giving the same answer $f(t) = 1 - e^{-t}$.

(3) (a) Again, there are many ways to do this.

First way: rememaber that a symmetric matrix is positive definite if and only if it can be written as $R^T R$ where R has linearly independent columns. In our case, let $c_1, \ldots, c_n$ be the diagonal entries of C and let B be the diagonal matrix with diagonal entries $\sqrt{c_1}, \ldots, \sqrt{c_n}$ (take the positive square roots). Then $C = B^T B$ and so $K = A^T B^T B A = (BA)^T (BA)$ so we take $R = BA$. Since A has linearly independent columns and B is invertible (because the c_i are nonzero numbers), we conclude that BA also has linearly independent columns.

Second way: Use the energy definition. Let x be a nonzero vector. We have to show that $x^T Kx > 0$. First, since A has linearly independent columns, this means that its null space is 0, so $Ax \neq 0$. Set $y = Ax$. Since C is diagonal and has positive diagonal entries, it is positive definite (this follows from the eigenvalue definition, or the submatrices definition, for example). So $y^T Cy > 0$, but $y^T Cy = x^T Kx$, so we're done.

(b) True, since A is diagonalizable with real eigenvalues, we can write $A = S\Lambda S^{-1}$ where Λ has real entries and the columns of S are some eigenvectors. Since we also know that A has orthonormal eigenvectors, we may choose these for S, and hence $S^{-1} = S^T$. But then $A = S\Lambda S^T$ and $A^T = (S^T)^T \Lambda^T S^T$. But $(S^T)^T = S$ for any matrix S, and $\Lambda^T = \Lambda$ because it is diagonal and square. So $A = A^T$ and A is symmetric.

Remark: Some people said that Hermitian matrices (topic not covered) are diagonalizable with orthonormal eigenvectors and have real eigenvalues but are not symmetric in general. This is true, and technically did not violate the directions of the problem since it did not specify that A has to have real entries, so received full credit.