1 (30 pts.)

(a) (25 pts.) Compute the determinant (as a function of x) of the 4×4 matrix

$$A = \begin{bmatrix} x & x & x & x \\ x & x & 0 & 0 \\ x & 0 & x & x \\ x & 0 & x & 1 \end{bmatrix}$$

(Note that all the entries x in the matrix represent the same number.)

There are several ways to do this. One way is to use the cofactor formula on the second row, which gives

$$-x * \det \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & x & 1 \end{bmatrix} + x * \det \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & 1 \end{bmatrix}.$$

The second determinant is zero because the first two rows are dependent, so we just need to expend the first. The first column is easy since there is only one nonzero element. Expanding gives $-x * x(x - x^2) = \boxed{x^4 - x^3}$.

(b) (5 pts.) Find all values of x for which A is singular.

A is singular exactly when the determinant is 0. This means $x^4 - x^3 = x^3(1 - x) = 0$, which means x is $\boxed{0 \text{ or } 1}$.

2 (35 pts.)

Let P_1 be the projection matrix onto the line through $(1, 1, 0)$ and P_2 is the projection matrix onto the line through $(0, 0, 1)$.

(a) (15 pts.) Compute $P = P_2P_1$. Note that there is a harder way and an easier way to perform this computation. Either way is valid. (The easier way uses associativity of matrix multiplication. Always a useful trick.)

We can calculate P_1 and P_2 directly via $P = A(A^TA)^{-1}A^T$ to obtain

$$P_2P_1 = B(B^TB)^{-1}B^TA(A^TA)^{-1}A^T,$$

where $B = [0, 0, 1]^T$ and $A = [1, 1, 0]^T$. However, in the middle of that mess is $B^TA = 0 * 1 + 0 * 1 + 1 * 0 = 0$, so the whole thing is the zero matrix.

The intutition behind this is that the two lines we are projecting onto are orthogonal. When we projected onto the first vector, we get some multiple of the first vector. Since this is orthogonal to the second vector, the second projection must get us 0.

(b) (5 pts.) Is $P = P_2P_1$ a projection matrix? (Explain simply.)

A matrix P is a projection matrix exactly when $P^2 = P$ and when P is symmetric, both of which are obviously true for the zero matrix (it is a projection onto the 0-dimensional space that has only the zero vector).

(c) (15 pts.) What are the four fundamental subspaces associated with P?

We have that $\boxed{C(P) = C(P^T) = 0}$, since both have column vectors that are all 0. We also have $\boxed{N(P) = N(P^T) = R^3}$ because every vector in R^3 gets sent to 0 via the zero matrix.

3 (35 pts.)

(a) (10 pts.) Perform Gram Schmidt on the two vectors $u = (1, 1, 1, 1)$ and $t = (t_1, t_2, t_3, t_4)$. The answer should be in the form q_1 and q_2, an orthonormal pair of vectors. You may wish to use the notation $\bar{t}$ for the mean of t, and $||t - \bar{t}u||$ for the norm of $t - \bar{t}u$.

For q_1, we take u and normalize by $||u|| = 2$, so dividing by the norm gives $\boxed{q_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}}$.

For q_2, we need to first subtract out the projection onto u (or q_1). This gives $t - (t^T u)u/(u^u) = t - (1/4)\sum t_i u = t - \bar{t}u$. Then we need to normalize, so the answer is $\boxed{q_2 = \frac{t - \bar{t}u}{||t - \bar{t}u||}}$.

(b) (10 pts.) Write a "QR" decomposition of $[u\ t]$, i.e. find a 4×2 matrix Q, and a 2×2 matrix R such $[u\ t] = QR$, where $Q^T Q = I$, and $R_{2,1} = 0$.

We already have Q by putting the two vectors from the previous part together, so

$$Q = \begin{bmatrix} 1/2 & \frac{t_1 - \bar{t}u}{||t - \bar{t}u||} \\ 1/2 & \frac{t_2 - \bar{t}u}{||t - \bar{t}u||} \\ 1/2 & \frac{t_3 - \bar{t}u}{||t - \bar{t}u||} \\ 1/2 & \frac{t_4 - \bar{t}u}{||t - \bar{t}u||} \end{bmatrix}.$$

We can get R by either multiplying $Q^T A$ or by eyeballing (since we know $QR = A$, it isn't hard to see, say, that getting all 1's in the first column needs 2 copies of the first column and none of the second, etc.). Multiplying $\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ \frac{t_1 - \bar{t}u}{||t - \bar{t}u||} & \frac{t_2 - \bar{t}u}{||t - \bar{t}u||} & \frac{t_3 - \bar{t}u}{||t - \bar{t}u||} & \frac{t_4 - \bar{t}u}{||t - \bar{t}u||} \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \\ 1 & t_4 \end{bmatrix}$

gives $\boxed{R = \begin{bmatrix} 2 & 2\bar{t} \\ 0 & \|t - \bar{t}u\| \end{bmatrix}}$.

(c)(5 pts.) When is R singular?

R is singular when its determinant is 0, so exactly when $\|t - \bar{t}u\| = 0$. This happens exactly when each $t_i = \bar{t}$, which happens when $\boxed{t = au}$ for some constant a (equivalently, $t_1 = t_2 = t_3 = t_4$).

(d)(10 pts) Use the QR decomposition of $A = [u\ t]$ (and not the normal equations with $A^T A$), to compute the slope of the best fit line $C + Dt$ to the data (t_i, b_i) for $i = 1, 2, 3, 4$. (In other words, compute a simple expression for D.)

The first key step is realizing that $\begin{bmatrix} C \\ D \end{bmatrix} = R^{-1}Q^T b$. One can derive this, say, via the original formula

$$\begin{aligned} x &= (A^T A)^{-1} A^T b && (1) \\ &= (R^T Q^T Q R)^{-1}(R^T Q^T) b && (2) \\ &= R^{-1} I (R^T)^{-1} R^T Q^T b && (3) \\ &= R^{-1} Q^T b. && (4) \end{aligned}$$

Now, a simplifying observation is that R^{-1} is still upper triangular, with the form $\frac{1}{2\|t-\bar{t}u\|}\begin{bmatrix} blah & blah \\ 0 & 2 \end{bmatrix}$. Since we only care about D, the second coordinate, we just need to multiply the lower right term by the second term of $Q^t b$, which is $q_2^T b$. So the answer is $\boxed{\dfrac{q_2^T b}{\|t-\bar{t}u\|} = \dfrac{(t-\bar{t}u)^T b}{\|t-\bar{t}u\|^2}}$.

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