18.06 Problem Set 9 Solutions

Problem 1: Do problem 15 from section 6.5.

Solution

Two ways to prove that $A+B$ is positive definite if A and B both are.

(i) As $x^TAx > 0$ and $x^TBx > 0$ for any x, we have $x^T(A+B)x = x^TAx+x^TBx > 0$.

(ii) As $A = R^TR$ for some R and $B = S^TS$ for some S,

$$A+B = R^TR + S^TS = (R^TS^T)\begin{pmatrix} R \\ S \end{pmatrix}(RS)^T\begin{pmatrix} R \\ S \end{pmatrix}$$

Problem 2: Do problem 4 section 6.6.

Solution

Any A with eigenvalues 0 and 1 are diagonalizable, hence is of the form $A = S\Lambda S^{-1}$. So any such A is similar to Λ.

Problem 3: Do problem 7 section 6.6.

Solution

(a)

$$x \in N(A) \Rightarrow Ax = 0 \Rightarrow M^{-1}AM(M^{-1}x) = M^{-1}Ax = 0 \Rightarrow M^{-1}x \in N(M^{-1}AM).$$

(b)

$$x \in N(M^{-1}AM) \Rightarrow M^{-1}AMx = 0 \Rightarrow AMx = 0 \Rightarrow Mx \in N(A).$$

So any vector in $N(A)$ (resp. $N(M^{-1}AM)$ is a linear combination of those in $N(M^{-1}AM)$ (resp. $N(A)$), hence is contained in it. That is, the two vector spaces consist of the same vectors.

Problem 4: Do problem 9 in section 6.7.

Solution

We consider the SVD backwards. Note

$$A = U\Sigma V^T \Rightarrow AV = U\Sigma$$

by multiplication V from the right on both sides, as V^TV is the identity. Take $U = (u_1, \cdots, u_n)$, $V = (v_1, \cdots, v_n)$ and Σ to be the $n \times n$ identity matrix. Then

$$A = UV^T \Rightarrow AV = U.$$

That is, $A = UV^T$ gives $Av_j = u_j$ for any $j = 1, \cdots, n$.

Problem 5: Do problem 13 in section 6.7.

Solution

As $A = QR$, we know the columns of Q are an orthonormal basis of $C(A)$ and R is upper triangular. Then we write the upper triangular R as $R = \Delta P^T$ by making the row vectors of R unit vectors. Thus Δ is diagonal with the diagonal elements lengths of the row vectors of R, and P has orthonormal columns. Therefore $A = Q\Delta P^T$ is a choice of SVD of A. In this sense, QR decomposition and SVD are the same.

However, we can also follow the standard method to get $A = U\Sigma V^T$, by computing $A^TAv_i = \sigma_i^2 v_i$ and taking $u_i = Av_i/\sigma_i$. However, the above SVD $A = Q\Delta P^T$ gives

$$A^TA = (Q\Delta P^T)^T Q\Delta P^T = P\Delta^2 P^T.$$

So the diagonal elements of Δ^2 are eigenvalues of A^TA, that is, σ_i^2's. Hence $\Delta = \Sigma$. In a word, the changes of Q may cause the change of U (hence that of V), but Σ does not change.

Problem 6: Do problem 4 in section 7.1. Solution

(a)

$$S(T(v)) = S(v) = v.$$

(b)

$$S(T(v_1 + v_2)) = S(T(v_1) + T(v_2)) = S(T(v_1)) + S(T(v_2)).$$

It is quadratic.

Problem 7: Do problem 8 in section 7.1.

Solution

(a) Range is the line $y = 0$, kernel is the line $x = y$ in the xy plane.
(b) Rang is the xy plane, kernel is the complementary line in $\mathbb{R}^3$.
(c) Range is the point $(0, 0)$, kernel is the plane.
(d) Range is the line $x = y$ in the xy plane, kernel is the line $x = 0$.

Problem 8: Do problem 13 in section 7.1.

Solution

The distribution law and the association law for multiplication give the linearity

$$A(cM + dN) = A(cM) + A(dN) = (Ac)M + (Ad)N = cA(M) + dA(N).$$