18.06 Problem Set 5 Solutions

Problem 1: Do problems 5 and 6 from section 4.2.

Solution

(5) For $\mathbf{a}_1 = (-1,2,2)$, the projection matrix is $P_1 = \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$.

For $\mathbf{a}_2 = (2,2,-1)$, the projection matrix is $P_2 = \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$.

We compute to see $P_1P_2 = 0$. This is because $\mathbf{a}_1$ and $\mathbf{a}_2$ are perpendicular. Precisely, if you project a vector $\mathbf{b}$ to $\mathbf{a}_1$, then it results a vector $\mathbf{p}$ on the same line as $\mathbf{a}_1$, whose projection to $\mathbf{a}_2$ is zero.

(6) In this case, $A = \begin{pmatrix} -1 & 2 \\ 2 & 2 \\ 2 & -1 \end{pmatrix}$. We compute $A^TA = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$, and $\mathbf{p} = A(A^TA)^{-1}A^T\mathbf{b} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{5} \\ 0 \end{pmatrix}$. The projection of $\mathbf{b}$ onto $\mathbf{a}_3$ is $\mathbf{p}_3 = P_3\mathbf{b} = \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix}$.

The projection $\mathbf{p}_1 = P_1\mathbf{b} = \begin{pmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{pmatrix}$.

The projection $\mathbf{p}_2 = P_2\mathbf{b} = \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{pmatrix}$.

So $\mathbf{p}_1 + \mathbf{p}_1 + \mathbf{p}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which is $\mathbf{b}$! The reason is that $\mathbf{a}_3$ is perpendicular to $\mathbf{a}_1$ and $\mathbf{a}_2$, hence when you compute the three projections of a vector and add them up you get back to the vector you start with.

Problem 2: Do problems 17 and 18 from section 4.2.

Solution

(17) Denote the number of columns of P by n. Apply $(I-P)^2$ to any vector $\mathbf{b}$ in $\mathbb{R}^n$, we have

$$(I-P)^2\mathbf{b} = (I-P)(b - P\mathbf{b}) = b - P\mathbf{b} - P\mathbf{b} + P^2\mathbf{b} = b - P\mathbf{b}$$

as $P^2\mathbf{b} = P\mathbf{b}$. That is,

$$(I-P)^2\mathbf{b} = (I-P)\mathbf{b}.$$

We take $\mathbf{b}$ to be $\mathbf{e}_i$, then the above equality means the i-th column of $(I-P)^2$ is equal to the i-th column of $(I-P)$. As i can be $1, \cdots, n$, we must have $(I-P)^2 = (I-P)$.

Note $(I-P)\mathbf{b} = b - P\mathbf{b}$. If $P\mathbf{b}$ is in the column space of A, then $b - P\mathbf{b}$ is a vector perpendicular to $C(A)$, hence is in the left nullspace of A. In a word, $I-P$ projects to the left nullspace.

(18)(a) This is left nullspace in $\mathbb{R}^2$ of the line through $(1,1)$, which is the line through $(1,-1)$.

(b) This is left nullspace in $\mathbb{R}^3$ of the line through $(1,1,1)$, which is the plane through $(1,-1,0)$ and $((0,1,-1))$.

Problem 3: Problem 2 section 4.3.

Solution

For $\mathbf{b}$, the linear system is

$$\begin{cases} C + D\cdot 0 = 0 \\ C + D\cdot 1 = 8 \\ C + D\cdot 3 = 8 \\ C + D\cdot 4 = 20 \end{cases}$$

For $\mathbf{p}$, the linear system is

$$\begin{cases} C + D\cdot 0 = 1 \\ C + D\cdot 1 = 5 \\ C + D\cdot 3 = 13 \\ C + D\cdot 4 = 17 \end{cases}$$

The solution is $C = 1, D = 4$.

Problem 4: Problem 12 section 4.3.

Solution

(a)

$$\hat{x} = \frac{a^T b}{a^t a} = \frac{b_1 + \cdots + b_m}{1 + \cdots + 1} = \frac{b_1 + \cdots + b_m}{m}.$$

(b) $e_i = b_i - (\mathbf{a}\hat{x})_i = b_i - \frac{\sum_{1\le i\le m} b_i}{m}$. So

$$\|\mathbf{e}\|^2 = \sum_{1\le i\le m} e_i^2 = \sum_{1\le i\le m} (b_i^2 - \frac{2b_i \sum_{1\le i\le m} b_i}{m} + \frac{(\sum_{1\le i\le m} b_i)^2}{m^2}$$

$$= \sum_{1\le i\le m} b_i^2 - \frac{2(\sum_{1\le i\le m} b_i)^2}{m} + \frac{(\sum_{1\le i\le m} b_i)^2}{m} = \sum_{1\le i\le m} b_i^2 - \frac{(\sum_{1\le i\le m} b_i)^2}{m}.$$

So

$$\|\mathbf{e}\| = \sqrt{\sum_{1\le i\le m} b_i^2 - \frac{(\sum_{1\le i\le m} b_i)^2}{m}}.$$

(c) In this case $\mathbf{e} = \mathbf{b} - \hat{\mathbf{b}} = (-2,-1,3)$. Easy to see it is perpendicular to $(3,3,3)$. The projection matrix is $(a^t a)^{-1} a a^t = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

Problem 5: Do problem 14 in section 4.3.

Solution

Note $\hat{x} = (A^T A)^{-1} A^T b$. We see

$$(A^T A)^{-1} A^T (b - Ax) = (A^T A)^{-1} A^T b - (A^T A)^{-1} A^T A x) = \hat{x} - x,$$

and

$$(b - Ax)^T A (A^T A)^{-1} = ((A^T A)^{-1} A^T (b - Ax))^T = (\hat{x} - x).$$

It is easy to see $A^T A)^{-1} A^T \sigma^2 I A (A^T A)^{-1} = \sigma^2 A^T A)^{-1} A^T A (A^T A)^{-1} = \sigma^2 (A^T A)^{-1}$.

Problem 6: Do problem 4 in section 4.4.

Solution

(a) $Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) Let one be $(0,0)$ and the other be $(0,1)$.

(c) Take the $\mathbf{B}$ to be $(1,-1,0)$ and $\mathbf{c} = (1,0,0)$, because it is clear $\mathbf{A} = (1,1,1)$ is perpendicular to $\mathbf{B}$ and $\mathbf{A}, \mathbf{B}, \mathbf{c}$ are linearly independent. Then we can find

$$\mathbf{C} = (1,1,-1).$$

Now take

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}}(1,-1,0), \quad \mathbf{q}_3 = \frac{1}{3\sqrt{2}}(1,-1,4).$$

Problem 7: Do problem 15 in section 4.4.

Solution

(a) Note $\mathbf{A} = (1, 2, -2)$ and $\mathbf{b} = (1, -1, 4)$. So

$$\mathbf{B} = \mathbf{b} - \frac{\mathbf{A}^T\mathbf{b}}{\mathbf{A}^T\mathbf{A}}\mathbf{A} = (2, 1, 2).$$

Take $\mathbf{c} = (0, 0, 1)$, which is easily to see to be linearly independent with $\mathbf{A}$ and $\mathbf{b}$. Then compute

$$\mathbf{C} = \mathbf{c} - \frac{\mathbf{A}^T\mathbf{c}}{\mathbf{A}^T\mathbf{A}}\mathbf{A} - \frac{\mathbf{B}^T\mathbf{c}}{\mathbf{B}^T\mathbf{c}}\mathbf{B} = (-9/2, 2/9, 1/9).$$

So we have

$$\begin{aligned} \mathbf{q}_1 &= \frac{1}{3}(1, 2, -2), \\ \mathbf{q}_2 &= \frac{1}{3}(2, 1, 2), \\ \mathbf{q}_3 &= \frac{1}{3}(-2, 2, 1). \end{aligned}$$

(b) Since $\mathbf{q}_2$ is perpendicular to the column space of A, it is in the left nullspace of A.

(c) Using part (a), we have

$$A = QR = \begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \\ -2/3 & 2/3 \end{pmatrix} \begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix},$$

so

$$\hat{x} = R^{-1}Q^T\mathbf{b} = \frac{1}{9}\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$