18.06 Problem Set 3 Solutions

Please note that the book problems listed below are out of the 4th edition. Please make sure to check that you are doing the correct problems.

Problem 1: Do problem 12 from section 3.3.(10pts)

Solution

If a matrix A has rank r, then the (dimension of the column space) = (dimension of the row space) = r. To find an invertible submatrix S, we need to find r linearly independent rows and r linearly independent columns. The choice of these columns and rows need not be unique - however the question suggests that we take the pivot rows and pivot columns.

For matrix A, after one row and column reduction:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

Here the $1st$ and $3rd$ columns are linearly independent, while the $1st$ and $2nd$ rows are also linearly independent. Hence rank $A = 2$. An appropriate submatrix is:

$$S_A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \tag{2}$$

Note also that

$$S_A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix} \tag{3}$$

would work, however

$$S_A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \tag{4}$$

would not! For matrix B:

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \tag{5}$$

Hence, rank $B = 1$:

$$S_B = \begin{pmatrix} 1 \end{pmatrix} \tag{6}$$

For matrix C, the rows and columns are already reduce - we only need to permute them to obtain echelon form.

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

rank $C = 2$, deleting the first column and middle row:

$$S_C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

Problem 2: Do problem 21 from section 3.4.(10pts)

$\boxed{\text{Solution}}$

a) $x + y + z = 4$ is equivalent to the matrix equation:

$$\begin{aligned} A\mathbf{x} &= 4 \quad (9) \\ A &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \quad (10) \end{aligned}$$

We seek the most general solution in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_n$ is a homogeneous solution to:

$$A\mathbf{x}_n = 0 \tag{11}$$

First note that rank A is $r = 1$, while the size of A is $m = 1$ and $n = 3$. Therefore A has infinitely many solutions. Moreover, there are $n - r = 2$ linearly independent homogeneous solutions. To find the general solution, parameterize the null space as follows. Let $y = c_1$, $z = c_2$ (we need two constants since the dimension of the null space is 2). Write $x = 4 - y - z$, so that $x = 4 - c_1 - c_2$. In vector form:

$$\mathbf{x} = \begin{pmatrix} 4 - c_1 - c_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{12}$$

We can check that this is a solution for any constants c_1 and c_2. Specifically:

$$\mathbf{x}_p = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \tag{13}$$

is a particular solution $A\mathbf{x}_p = 4$, while

$$\mathbf{x}_n = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{14}$$

is the homogeneous solution $A\mathbf{x}_n = 0$ (Note that the null space of A is spanned by the two linearly independent vectors $(-1, 1, 0)^T$ and $(-1, 0, 1)^T$)

b) We have:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \tag{15}$$

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{16}$$

Row reduce both sides:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{array}\right) \tag{17}$$

$$\tag{18}$$

From the row reduction, we see A has two linearly independent rows and columns so that rank $r = 2$, $n = 2$ and $m = 3$. The second equation implies $y = 0$. The first equation is under-determined corresponding with the fact that the null space of A has dimension $m - r = 1$. To parameterize the solutions let $z = c_1$ so that $x = 4 - z = 4 - c_1$. The solution is:

$$\mathbf{x} = \begin{pmatrix} 4 - c_1 \\ 0 \\ c_1 \end{pmatrix} \tag{19}$$

$$= \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{20}$$

or

$$\mathbf{x}_p = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \tag{21}$$

$$\mathbf{x}_h = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{22}$$

Problem 3: When we perform elimination via row operations on a matrix A to end up with an upper triangular matrix U, we know this operation can be expressed as multiplication by an elimination matrix E so that $EA = U$. Why does E change the column space of A but not the nullspace? Give an example of A such that EA has a different column space. Explain why the nullspace of A and U are the same. (10pts)

Solution

The product $y = Ax$ (for all vectors x) defines the column space of A. Meanwhile the null space of A is any vector x such that $Ax = 0$. Multiplying by an elimination matrix E changes the column space (when rank A < n = size of columns), but not the null space.

Multiplying by E acts on the rows of A. If rank A is < size of the columns, then row reduction yields a zero row. A zero row implies the column space will never have components in the row with a zero. The original column space matrix product Ax could however have nonzero elements in any given component (Precisely, the preservation of the column space by a matrix E implies that for every vector x_1, there is a vector x_2 such that $Ax_2 = EAx_1$. When A is not invertible (or more precisely rank A < n = size of columns) this is false).

Multiplication by E preserves the null space of A. E is the product of many elimination matrices - each of which is invertible. Hence, E is invertible. If $Ax = 0$, then certainly $E(Ax) = E0 = 0$. Meanwhile if $EAx = 0$, then since E^{-1} exists, it follows that $Ax = 0$.

An example of A such that EA has a different column space:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \tag{23}$$

take

$$E = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \tag{24}$$

so that

$$EA = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \tag{25}$$

The column space of A is:

$$C(A) = c \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{26}$$

while the column space of EA is:

$$C(EA) = c\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{27}$$

Here c is an arbitrary constant.

A and U have the same null space. We can row reduce A by the product of many elimination matrices E so that $EA = U$. Since each elimination matrix is invertible, then the product E is also invertible. From an identical argument made above, it follows that $Ux = 0 \iff Ax = 0$.

Problem 4: Suppose that a is a real, nonzero number. Consider the matrix

$$\begin{pmatrix} a & b & c & c \\ a & a & b & c \\ a & a & a & b \\ a & a & a & a \end{pmatrix}.$$

Find the row reduced echelon form and the nullspace of A when:

(a) $a \neq b$ (3pts)

(b) $a = b$ and $b \neq c$ (4pts)

(c) $a = b = c$. (3pts)

$\boxed{\text{Solution}}$ Reduce the matrix into upper triangular form:

$$\begin{pmatrix} a & b & c & c \\ a & a & b & c \\ a & a & a & b \\ a & a & a & a \end{pmatrix} \sim \begin{pmatrix} a & b & c & c \\ 0 & a-b & b-c & 0 \\ 0 & a-b & a-c & b-c \\ 0 & a-b & a-c & a-c \end{pmatrix} \tag{28}$$

$$\sim \begin{pmatrix} a & b & c & c \\ 0 & a-b & b-c & 0 \\ 0 & 0 & a-b & b-c \\ 0 & 0 & a-b & a-c \end{pmatrix} \sim \begin{pmatrix} a & b & c & c \\ 0 & a-b & b-c & 0 \\ 0 & 0 & a-b & b-c \\ 0 & 0 & 0 & a-b \end{pmatrix} \tag{29}$$

a) When $a \neq b$, the matrix has echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{30}$$

so that the null space is just $\mathbf{x} = 0$.

b) When $a = b$, $b \neq c$ the matrix has echelon form:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{31}$$

The null space has dimension 1:

$$\mathbf{x} = c \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{32}$$

c) When $a = b = c$ the matrix has echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{33}$$

The null space has dimension 3:

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{34}$$

Problem 5: (MATLAB problem!)

(a) Define a 4 by 4 random matrix A and column vector b:
`A= rand(4); b = [1;1;1;1];` Solve the equation Ay=b by typing:
`y = A \ b` Type: `A*y - b` What do you get? (2pts)

(b) Write a program to define a matrix A such that $A_{i,j} = e^{15(i+j)}$. To do this we write:

```
for i=1:4
    for j=1:4
        A(i,j)=exp(15*(i+j));
    end
end
```

```
A=A.*rand(4);
```

```
b=[1;1;1;1];
```

The command A.$*$ rand(4) multiplies each entry in A termwise by a random number. We do this so that our matrix A is not so singluar. Now, type: `x = A \ b`. What is $Ax - b$? (3pts)

(c) In part (b), why is $Ax - b$ so far from 0? Notice that $Ax - b$ should be equal to 0, because x is supposed to be a solution to the equation $Ax = b$. Hint: In essence, imagine that to solve for x, MATLAB is row reducing. Note that in MATLAB $10^{-17} = 0$ because MATLAB only sees 16 significant figures. (Type: `format long` and look at 10^{-16} and compare to 10^{-17}.) (5pts)

$\boxed{\text{Solution}}$ a) Hopefully, you get 0.

b) You'll find $Ax - b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \text{something large and nonzero.} \end{pmatrix}$

c) The smallest element of A is $A(1,1) \sim 10^{13}$, while $A(4,4) \sim 10^{51}$. When performing floating point arithmetic, MATLAB only keeps 16 decimals (note that when inverting a random matrix, MATLAB will construct an LU factorization followed by performing a sequence of forward and back substitutions). During the inversion process, the 1's on the RHS, will be divided by numbers exceeding 10^{16}. Consequently, MATLAB effectively solves $Ax = 0$ and not $Ax = b$.

Problem 6: Do problem 35 from section 3.2 in the book.

Solution

Expand out in block form. Let $\mathbf{u} = [\mathbf{x}; \mathbf{y}]^T$ be an 8 element column vector where $\mathbf{x}$ and $\mathbf{y}$ are 4 element column vectors. Then $B = [A; A]$ multiplies out:

$$B\mathbf{u} = A\mathbf{x} + A\mathbf{y} \tag{35}$$

For $\mathbf{u} \in N(B)$:

$$\begin{aligned} B\mathbf{u} &= 0 \quad (36)\\ A\mathbf{x} + A\mathbf{y} &= 0 \quad (37)\\ A^{-1}(A\mathbf{x} + A\mathbf{y}) &= 0 \quad (38)\\ \mathbf{x} &= -\mathbf{y} \quad (39) \end{aligned}$$

Hence, the null space is described by:

$$\mathbf{u} = c_1 \begin{pmatrix} \mathbf{x}_1 \\ -\mathbf{x}_1 \end{pmatrix} + c_2 \begin{pmatrix} \mathbf{x}_2 \\ -\mathbf{x}_2 \end{pmatrix} + c_3 \begin{pmatrix} \mathbf{x}_3 \\ -\mathbf{x}_3 \end{pmatrix} + c_4 \begin{pmatrix} \mathbf{x}_4 \\ -\mathbf{x}_4 \end{pmatrix} \tag{40}$$

where the jth element of $\mathbf{x}_i = 1$ if $j = i$ and 0 if $j \neq i$. One can also write:

$$\mathbf{u} = \begin{pmatrix} I \\ -I \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{41}$$

where I is the 4×4 identity matrix.

Problem 7:

1. Suppose A is a matrix such that $A^2 = A$. (5pts) True or False
   (a) $N(A)$ is a subspace of $C(A)$.
   (b) $C(A)$ is a subspace of $N(A)$.
   (c) The only vector common to both spaces is 0.
2. Suppose A is a matrix such that $A^2 = 0$. (5pts) True or False
   (a) $N(A)$ is a subspace of $C(A)$.
   (b) $C(A)$ is a subspace of $N(A)$.

(c) The only vector common to both spaces is 0.

Be sure to justify your answers and explain why each option is true, or give an example of how it fails.
$\boxed{\text{Solution}}$ Question 1)

1. $N(A)$ is a subspace of $C(A)$ - False. Take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{42}$$

The null space is:

$$N(A) = c \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{43}$$

The column space is:

$$C(A) = c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{44}$$

2. $C(A)$ is a subspace of $N(A)$ - False (same example as above).

3. The only vector common to both spaces is 0 - True. Matrices $A^2 = A$ are called projectors. They are special in the sense that they project a vector x onto the column space of A. Explicitly, we can write $B = I - A$. Then $B^2 = B$ (is also a projector), $AB = BA = 0$ and $A + B = I$. Given any vector x, we may now uniquely decompose $x = u + v$, where $u = Ax$ and $v = Bx$. Now suppose $x \in C(A)$. Then $x = Ay$ for some y, which means $v = Bx = BAy = 0$ and so $u = x$. Meanwhile if $x \in N(A)$ then $Ax = 0$ so that $u = 0$ and $v = x$. Therefore, if $x \in N(A)$ and $x \in C(A)$, it follows that $u = v = 0$, so that $x = u + v = 0$. Note that for a projector A, all vectors x decompose into a piece $u \in C(A)$ and a piece $v \in N(A)$.

Question 2)

1. $N(A)$ is a subspace of $C(A)$ - False. Take

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{45}$$

The null space is:

$$N(A) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{46}$$

The column space is:

$$C(A) = c \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{47}$$

2. $C(A)$ is a subspace of $N(A)$ - True. If $x \in C(A)$ then $x = Ay$ for some y. It follows that $Ax = A^2y = 0$, so that $x \in N(A)$.

3. The only vector common to both spaces is 0 - False. This follows from property 2) above with an explicit construction in 1).