18.06 Problem Set 2 Solutions

Problem 1: Do problem 27 from section 2.5 in the book.
Solution (8pts)

$$\begin{pmatrix}1&0&0&1&0&0\\2&1&3&0&1&0\\0&0&1&0&0&1\end{pmatrix} \to \begin{pmatrix}1&0&0&1&0&0\\0&1&3&-2&1&0\\0&0&1&0&0&1\end{pmatrix} \to \begin{pmatrix}1&0&0&1&0&0\\0&1&0&-2&1&-3\\0&0&1&0&0&1\end{pmatrix}.$$

Hence for $A = \begin{pmatrix}1&0&0\\2&1&3\\0&0&1\end{pmatrix}$, $A^{-1} = \begin{pmatrix}1&0&0\\-2&1&-3\\0&0&1\end{pmatrix}$.

And

$$\begin{pmatrix}1&1&1&1&0&0\\1&2&2&0&1&0\\1&2&3&0&0&1\end{pmatrix} \to \begin{pmatrix}1&1&1&1&0&0\\0&1&1&-1&1&0\\0&1&2&-1&0&1\end{pmatrix} \to \begin{pmatrix}1&1&1&1&0&0\\0&1&1&-1&1&0\\0&0&1&0&-1&1\end{pmatrix}$$

$$\to \begin{pmatrix}1&1&1&1&0&0\\0&1&0&-1&2&-1\\0&0&1&0&-1&1\end{pmatrix} \to \begin{pmatrix}1&0&0&2&-1&0\\0&1&0&-1&2&-1\\0&0&1&0&-1&1\end{pmatrix}.$$

Hence for $A = \begin{pmatrix}1&1&1\\1&2&2\\1&2&3\end{pmatrix}$, $A^{-1} = \begin{pmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{pmatrix}$.

Problem 2: Do problem 13 from section 2.6.
Solution (10pts) The required eliminations are, 1) subtracting the first row from the other rows, 2) subtracting the second row from the third and fourth, and 3) subtracting the third row from the fourth. The resulting matrix U is

$$\begin{pmatrix}a&a&a&a\\0&b-a&b-a&b-a\\0&0&c-b&c-b\\0&0&0&d-c\end{pmatrix}.$$

Hence, $A = \begin{pmatrix}a&a&a&a\\a&b&b&b\\a&b&c&c\\a&b&c&d\end{pmatrix} = \begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}U$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{pmatrix} = LU.$$

To make it have four pivots, we need $a \neq 0, b \neq a, c \neq b, d \neq c$.

Problem 3: Do problem 15 from section 2.6.

Solution (10pts) From $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \end{pmatrix}$, we easily get $c_1 = 2, c_2 = 3$. Also $\begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, hence $x_1 = -5, x_2 = 3$. $\mathbf{x} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$.

On the other hand, $A = LU = \begin{pmatrix} 2 & 4 \\ 8 & 17 \end{pmatrix}$, and solving the augmented matrix $\begin{pmatrix} 2 & 4 & 2 \\ 8 & 17 & 11 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 4 & 2 \\ 0 & 1 & 3 \end{pmatrix}$ gives us the same answer $\mathbf{x} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$. We see that $\mathbf{c}$ appears as the right column of the last step of the augmented matrix calculation.

Problem 4: Do problem 12 from section 2.7.

Solution (10pts) P exchanges the coordinates of $\mathbf{x}$ (and $\mathbf{y}$). If $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, then $P\mathbf{x} = \begin{pmatrix} x_{\sigma(1)} \\ \vdots \\ x_{\sigma(n)} \end{pmatrix}, P\mathbf{y} = \begin{pmatrix} y_{\sigma(1)} \\ \vdots \\ y_{\sigma(n)} \end{pmatrix}$ for some permutation σ. Hence $\mathbf{x} \cdot \mathbf{y} = \sum x_i y_i = \sum x_{\sigma(i)} y_{\sigma(i)} = P\mathbf{x} \cdot P\mathbf{y}$. Now, $\mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y} = P\mathbf{x} \cdot P\mathbf{y} = (P\mathbf{x})^T P\mathbf{y} = \mathbf{x}^T(P^TP)\mathbf{y}$ for any $\mathbf{x}, \mathbf{y}$. Let $\mathbf{e_i}$ be the coordinate vector that has 1 at ith coordinate and 0's elsewhere. Choose $\mathbf{x} = \mathbf{e_i}, \mathbf{y} = \mathbf{e_j}$. Then $\mathbf{x}^T(P^TP)\mathbf{y}$ is the (i, j)-entry of P^TP, and $\mathbf{x}^T\mathbf{y} = \delta_{ij}$. Hence P^TP is the identity matrix.

Let $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Then $P\mathbf{x} \cdot \mathbf{y} = 11$ while $\mathbf{x} \cdot P\mathbf{y} = 16$.

Problem 5: Do problem 17 from section 2.7.

Solution (10pts distributed as follows)(a)(3pts) Any symmetric matrix with determinant 0. e.g. $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ldots$

(b) (3pts) Any invertible symmetric matrix with the first entry 0. e.g. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots$

(c) (4pts) Symmetric matrix with nonzero first entry and at least one negative pivot.
e.g. $\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Problem 6: Do problem 10 from section 3.1.

Solution (12pts) (a)(2pts) Subspace, because

$$c(x, x, z) + d(x', x', z') = (cx + dx', cx + dx', cz + dz').$$

(b) (2pts) Not a subspace. (1,0,0)+(1,0,0)=(2,0,0).
(c) (2pts) Not a subspace. (1,0,0)+(0,1,1)=(1,1,1).
(d) (2pts) Subspace, by definition of linear combination.
(e) (2pts) Subspace, these are the vectors orthogonal to (1,1,1).
(f) (2pts) Not a subspace. $(-1) \times (1, 2, 3) = (-1, -2, -3)$.

Problem 7: Do problem 20 in section 3.1.

Solution (a) (5pts) The column space consists of the vectors $\begin{pmatrix} x_1 + 4x_2 + 2x_3 \\ 2x_1 + 8x_2 + 4x_3 \\ -x_1 - 4x_2 - 2x_3 \end{pmatrix}$.
Let's substitute $x_1 + 4x_2 + 2x_3$ with a new variable z. Then the column space consists of the vectors $\begin{pmatrix} z \\ 2z \\ -z \end{pmatrix} = z \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. Hence they are scalar multiples of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(b) (5pts) Similarly, by substituting $x_1 + 4x_2$ with a new variable z, we see that the column space consists of the vectors $\begin{pmatrix} x_1 + 4x_2 \\ 2x_1 + 9x_2 \\ -x_1 - 4x_2 \end{pmatrix} = z \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Hence they are linear combinations of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.