18.06 Problem Set 1 Solutions

Problem 1: Do problem 27 from section 1.2 in the book.
Solution (10pts)
$||v-w|| \leq ||v||+||w|| = 5+3 = 8$ and $||v-w|| \geq ||v||-||w|| = 5-3 = 2$. (5pts)
$|v \cdot w| = ||v|| \cdot ||w|| \cos\theta \leq ||v|| \cdot ||w||$
Thus we find that $-||v||||w|| \leq |v \cdot w| \leq ||v|| \cdot ||w||$. Thus the minimum value occurs when the dot product is a small as possible: ie. v and w are parrallel, but point in opposite directions. So smallest value is -15. The maximum value occurs when the dot product is as large as possible, thus occurs when v and w are parallel and point in the same direction. Thus the largest value is 15. (5pts)

Problem 2: Do problem 8 from section 2.1.
Solution (10pts)
Normally 4 "planes" in 4-dimensional space meet at a point (2pts). Normally 4 column vectors in a 4-dimensional space can combine to produce **b**.
The combination of the 4 column vectors producing **b** is:

$$1\begin{pmatrix}1\\1\\1\\0\end{pmatrix}+2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\3\\2\end{pmatrix}\text{ (4pts)}$$

The system of linear equations this is satisfying is (4pts):

$$\begin{aligned}x+y+z+t&=3\\y+z+t&=3\\z+t&=3\\t&=2.\end{aligned}$$

Problem 3: Do problem 11 from section 2.2.
Solution (10pts)
(a) (5pts) Suppose a system of linear equations has 2 distinct solutions $\mathbf{x}$ and $\mathbf{y}$ both satisfying $A\mathbf{x}=\mathbf{b}$ and $A\mathbf{y}=\mathbf{b}$. Then $A(\mathbf{x}-\mathbf{y})=0$, so in particular, given any real

number t, $At(\mathbf{x}-\mathbf{y})=0$. Thus any vector of the form $\mathbf{x}+t(\mathbf{x}-\mathbf{y})$ solves the linear system since

$$\begin{aligned} A(\mathbf{x}+t(\mathbf{x}-\mathbf{y})) &= A\mathbf{x}+At(\mathbf{x}-\mathbf{y}) \\ &= \mathbf{b}+0 \end{aligned}$$

. Since by hypothesis **x-y** is nonzero, there are infinitely many solutions to this system corresponding to the line $\mathbf{x}+t(\mathbf{x}-\mathbf{y})$. (b) (5pts) If 25 planes meet at two points, the also meet in the line that passes through both of these points.

Problem 4: Do problem 21 from section 2.2.
Solution (10pts)
Begin by row reducing the augmented matrix $A|b$ (or $K|b$.) (5pts each)

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 1.5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix}.$$

Thus the pivots are the diagonal entries, and the solution is $\begin{pmatrix} -1 \\ 2 \\ -3 \\ 4 \end{pmatrix}$

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & 1.5 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix}.$$

Thus this matrix has the same pivots, and the solution is $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

Problem 5: Do problem 14 from section 2.3.
Solution (10pts)
Observe that these are the elimination matrices corresponding to the row reduction performed in the previous problem part 2. Thus these elimination matrices are:

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$E_{32} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$E_{43} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{pmatrix}.$$

Problem 6: Do problem 23 from section 2.4.
Solution (10pts)
(a)(5pts) A nonzero matrix A such that $A^2 = 0$ is :

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(b)(5pts) We use a 3 by 3 matrix. We want A such that $A^2 \neq 0$ but $A^3 = 0$. For example, try

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

You can check that $A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $A^3 = 0$.

Problem 7: Do problem 29 in section 2.5
Solution (10pts)

(a)(3pts) **T** A 4 by 4 matrix with a row of zeros can not be invertible because it can have at most 3 pivots.
(b)(3pts) **F** To justify, must give a counter example. Consider for example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

This matrix has 1's along the main diagonal, but only has 2 pivots, thus is not invertible.
(c)(4pts) **T** If A is invertible, then necessarily A^{-1} is invertible with inverse A. If A^{-1} were not invertible, there would be a nonzero vector $\mathbf{x}$ such that $A^{-1}\mathbf{x} = 0$. But then

$$\mathbf{x} = I\mathbf{x} = AA^{-1}\mathbf{x} = A0 = 0,$$

which contradicts our assumption that $\mathbf{x}$ was nonzero. Thus A^{-1} is invertible.

Similarly, suppose there were a nonzero $\mathbf{x}$ such that $A^2\mathbf{x} = 0$, then by an analogous argument, we see that

$$A^{-1}A^2\mathbf{x} = (A^{-1}A)A\mathbf{x} = A\mathbf{x} = 0.$$

Since A is invertible, this can only be true if $\mathbf{x}$ was zero to begin with. Thus A^2 must also be invertible.