SOLUTIONS TO QUIZ 3

Problem 1. (6 points each)

$A = \begin{pmatrix} a & c+di \\ c-di & b \end{pmatrix}$

a) This matrix is clearly hermetian.

b) Thus, the two eigenvalues are real.

c) The sum of the eigenvalues is $tr(A) = a+b$.

d) The product of the eigenvalues is $det(A) = ab-(c+di)(c-di) = ab-(c^2+d^2)$.

e) We need to solve $\begin{pmatrix} a-\lambda & c+di \\ c-di & b-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$. We see that $\begin{pmatrix} -(c+di) \\ a-\lambda \end{pmatrix}$ is one such (note that this solves the top row equation, and the other by singularity of the matrix).

Problem 2. (8 points each)

$A = \begin{pmatrix} x & 3/5 \\ y & z \end{pmatrix}$.

a) A is positive definite if $x > 0$ and $det(A) = xz - 3y/5 > 0$.

b) A is Markov if $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x+y = 1$ and $z = 2/5$.

c) A is singular if $0 = det(A) = xz - 3y/5$.

d) Well, we need $(3/5)^2 + z^2 = 1$, so $z^2 = 16/25$, so set $z = 4/5$. So set $x = -4/5$ and $y = 3/5$. Flip the signs around to get the other possibilities.

Problem 3. (13 points)

As F has four distinct eigenvlaues, it is diagonalizable, i.e., $F = S\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix}S^{-1}$. Thus $F^4 = S\begin{pmatrix} (-2)^4 & 0 & 0 & 0 \\ 0 & 2^4 & 0 & 0 \\ 0 & 0 & (2i)^4 & 0 \\ 0 & 0 & 0 & (-2i)^4 \end{pmatrix}S^{-1} = S\begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}S^{-1} = \begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}$.

As this is already a Jordan matrix, this is the Jordan form of F^4. The underlying reason is that F is diagonalizable, hence $\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix}$ is its Jordan form.

Problem 4. (25 points)

$A = \begin{pmatrix} x & ? \\ ? & ? \end{pmatrix}$. As A is supposed to be Markov, we must have $A = \begin{pmatrix} x & y \\ 1-x & 1-y \end{pmatrix}$, and A is singular implies $x(1-y) - y(1-x) = 0$, therefore $0 = x - xy - y + xy = x - y$, so $y = x$. Thus $A = \begin{pmatrix} x & x \\ 1-x & 1-x \end{pmatrix}$. As A is Markov, we know that $\lambda_1 = 1$ is an eigenvalue. As A is singular, we know that the product of the eigenvalues is 0. Therefore, $\lambda_2 = 0$ is another eigenvalue, and so A is diagonalizable; $A = S\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}S^{-1}$. Thus $A^{2008} = S\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{2008}S^{-1} = S\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}S^{-1} = A$.