QUIZ 2 SOLUTIONS

1. (10 points). $det(-A^t) = (-1)^{1000} det(A^t) = det(A)$.

2. a) (10 points). The projection matrix of a matrix A is $P = A(A^tA)^{-1}A^t$. So the projection matrix of QA is $(QA)(A^tQ^tQA)^{-1}A^tQ^t = QA(A^tA)^{-1}A^tQ^t = QPQ^t$ where we have used that $Q^tQ = I$.

b) (10 points). By defintion $c - Pc$ is orthogonal to the space $span\{a,b\} = span\{q_1,q_2\}$. So we can choose $q_3 = (c-Pc)/||c-Pc||$.

c) (10 points). Let s_1,s_2,s_3 denote the rows of Q, and r_1,r_2,r_3 denote the columns of R. Then $c = \begin{pmatrix} s_1 \cdot r_3 \\ s_2 \cdot r_3 \\ s_3 \cdot r_3 \end{pmatrix} = Qr_3$. Since orthogonal matrices preserve the lengths of vectors, this implies $||c|| = ||r_3||$.

3. (15 points). Well, the matrix $uu^t = \left(u_iu_j\right)_{1\le i,j\le n}$; so $I + tuu^t = (\delta_{ij} + tu_iu_j) = Q$. In particular, this matrix is symmetric, so the orthogonality condition reduces to $Q^2 = I$. Writing this condition out gives $I = (I+tuu^t)(I+tuu^t) = I + 2tuu^t + t^2(uu^t)^2$, or equivalently $t(2uu^t + t(uu^t)^2) = 0$. But now we have that $(uu^t)^2 = u(u^tu)u^t = uu^t$ because $u^tu = 1$ (u has length 1). So our equation becomes $t(2+t)(uu^t) = 0$. Clearly $t = 0$ and $t = -2$ are the solutions.

4. (15 points). We have that $C + Dt + (1-E)t = (C+E) + (D-E)t$. Thus we see that E is a free variable: it is not uniquely determined, and in fact can take any value. Given this, just write down the usual least squares equations but treat $C+E$ and $D-E$ as your variables: the matrix A has two columns: the first consists of n 1's, the second is the vector (t_i). Then solve $A^tA\begin{pmatrix} C+E \\ D-E \end{pmatrix} = A^tb$.

5. a) (15 points). Yes. As A is invertible, it's column space is the full space $\mathbb{R}^n$. The same is true of A^{-1}.

b) (15 points). No. consider $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It has nonzero column space, but its square is 0.