SOLUTIONS TO PSET 8

Problem 1. (5 points each)

1. We carry out the three steps on page 306. Firstly the eigenvalues of $\begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$ are clearly 4 and 1. Thus, the eigenvectors can be found by solving $\begin{pmatrix} 0 & 3 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ and $\begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$, yielding $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Now, $\mathbf{u}(0) = \begin{pmatrix} 5 \\ -2 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus our solution is $\mathbf{u}(t) = 3\mathbf{e}^{4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\mathbf{e}^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

2. The general solution to $z' = z$ is $z = c\mathbf{e}^t$, so $z(0) = -2$ implies that $c = -2$. So we now solve $y' = 4y - 6\mathbf{e}^t$. The reader can verify that $3\mathbf{e}^{4t} + 2\mathbf{e}^t$ works.

Problem 2. (5 points each)

a) $2u_1u_1' + 2u_2u_2' + 2u_3u_3' = 2[u_1(cu_2 - bu_3) + u_2(au_3 - cu_1) + u_3(bu_1 - au_2)] = 0$. So $||\mathbf{u}(t)||^2$ has derivative identically zero, and hence is a constant function.

b) Suppose $A^t = -A$, and let $Q = \mathbf{e}^{At}$. Then $Q^t = (\mathbf{e}^{At})^t = (1 + At + A^2t^2/2 + ...)^t = (1 + A^t t + (A^t)^2t^2/2 + ...) = (1 + (-A)t + (-A)^2t^2/2 + ...) = \mathbf{e}^{-At}$. Thus $Q^t = Q^{-1}$ as required.

Problem 3. (5 points each)

a) $(\mathbf{e}^{At})^{-1} = \mathbf{e}^{-At}$ since $\mathbf{e}^{At}\mathbf{e}^{-At} = \mathbf{e}^{At-At} = I$.

b) Suppose $A\mathbf{x} = \lambda\mathbf{x}$. Then $\mathbf{e}^{At}\mathbf{x} = (1 + At + A^2t^2/2 + ...)\mathbf{x} = \mathbf{x} + \lambda t\mathbf{x} + (\lambda^2t^2/2)\mathbf{x} + ... = (1 + \lambda t + (\lambda t)^2/2 + ...)\mathbf{x} = \mathbf{e}^{\lambda t}\mathbf{x}$. Further, $\mathbf{e}^{\lambda t}$ is never zero because $\mathbf{e}^y$ is never zero, for any y. Finally, since $det(\mathbf{e}^{At})$ is the product of the eigenvalues, this implies that $\mathbf{e}^{At}$ is invertible.

Problem 4. (2.5 points each)

a) False. Consider $A = \begin{pmatrix} 1 & 7 \\ 0 & 2 \end{pmatrix}$.

b) True, if one reads the question to mean that A has distinct eigenvectors (if you don't assume this, take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ as a counterexample). If we do assume this, we can write $A = S\Lambda S^{-1}$ where Λ is diagonal (the matrix of eigenvalues) and S is the matrix of eigenvectors. Since the eigenvectors are orthogonal, we can scale S to be an orthonormal matrix (eigenvalues aren't affected by scaling). Then $S^{-1} = S^t$ and so $A^t = (S^t)^{-1}\Lambda S^t = S\Lambda S^{-1} = A$.

c) True. Let $A^t = A$, and let $B = A^{-1}$ so that $AB = BA = I$. Then transposing these equations gives $I = B^tA^t = B^tA$ and $I = A^tB^t = AB^t$, thus $B^t = A^{-1}$ as well, so $B^t = B$.

d) False.

Problem 5. Well, $det\begin{pmatrix} 2-\lambda & b \\ 1 & -\lambda \end{pmatrix} = -(\lambda)(2-\lambda) - b = \lambda^2 - 2\lambda - b$, so the eigenvalues are $1 + \sqrt{1+b}$ and $1 - \sqrt{1+b}$ by the quadratic formula. The eigenvectors therefore are the solutions to $\begin{pmatrix} 1-\sqrt{1+b} & b \\ 1 & -1-\sqrt{1+b} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ and $\begin{pmatrix} 1+\sqrt{1+b} & b \\ 1 & -1+\sqrt{1+b} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$, which are $\begin{pmatrix} 1+\sqrt{1+b} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1-\sqrt{1+b} \\ 1 \end{pmatrix}$, respectively. These vectors are orthogonal when $b = 1$, this makes $A = Q\Lambda Q^t$ possible. When $b = -1$, the two eigenvectors become equal, and then the $S\Lambda S^{-1}$ factorization is impossible. As $det(A) = -b$, setting $b = 0$ makes inverting A impossible.

Problem 6. (5 points each)

a) $(A^H A)^H = A^H (A^H)^H = A^H A$, so $A^H A$ is hermetian.

b) $A^H A\mathbf{z} = 0$ implies that $0 = \mathbf{z}^H A^H A\mathbf{z} = (A\mathbf{z})^H A\mathbf{z} = ||A\mathbf{z}||^2$; thus $A\mathbf{z} = 0$. So $N(A) = N(A^H A)$, and $A^H A$ is invertible iff $N(A) = 0$.

Problem 7. (5 points)

1. Well, U is unitary implies $U^{-1} = U^H$, so $(U^{-1})^H = (U^H)^{-1} = (U^{-1})^{-1} = U$, so U^{-1} is also unitary. Further $(UV)^H UV = V^H U^H UV = V^H V = I$, so UV is unitary.

2. Well, we know that for any complex matrix A, $det(A^H) = det(\bar{A}^t) = \bar{det}(A^t) = \bar{det}(A)$. Now, if A is hermetian, $A^H = A$ implies $det(A^H) = det(A)$, so $\bar{det}(A) = det(A)$, and $det(A)$ is real.

Problem 8. The eigenvalues are the roots of $det\begin{pmatrix} 2-\lambda & 1-i \\ 1+i & 3-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda-4)(\lambda-1)$, so 4 and 1 fit the bill. The eigenvectors are the solutions to $\begin{pmatrix} -2 & 1-i \\ 1+i & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ and $\begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$, which are $\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$ and $\begin{pmatrix} -1+i \\ 1 \end{pmatrix}$ repectively. Thus the diagonalization is $A = \begin{pmatrix} 1 & -1+i \\ 1+i & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/3 & (1-i)/3 \\ (-1-i)/3 & 1/3 \end{pmatrix}$.

Problem 9. (5 points each)

1. We have $F_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The inverse of the third factor is itself, as it is a symmetric permutation matrix. The second factor can be inverted by inverting each of its constituent 2×2 matrices, and the first factor can be inverted by inspection to get: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2i & 0 & -1/2i \end{pmatrix}$.

2. Transposing the equation gives: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & i & 0 & -i \end{pmatrix}$.