SOLUTIONS TO PSET 7

1. (5 points each)

1. $det(A-\lambda I) = det\begin{pmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix} = (-\lambda)(3-\lambda)-4 = \lambda^2-3\lambda-4 = (\lambda-4)(\lambda+1)$; so the eigenvalues are 4 and -1. So, to get the eigenvector corresponding to 4, we solve $\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ and arrive at $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. For the -1 eigenvector, we solve $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ and get $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Next, $det(A^{-1}-\lambda I) = det\begin{pmatrix} -3/4-\lambda & 1/2 \\ 1/2 & -\lambda \end{pmatrix} = (-\lambda)(-3/4-\lambda)-1/4 = \lambda^2+(3/4)\lambda-1/4 = (\lambda+1)(\lambda-1/4)$; so we get -1 and $1/4$, the inverses of the above. The eigenvectors have to be the same, because if we have $A\mathbf{x} = \lambda\mathbf{x}$, then we can apply A^{-1} to both sides to get $\mathbf{x} = \lambda A^{-1}\mathbf{x}$, and then $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$.

2. $det(A-\lambda I) = det\begin{pmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{pmatrix} = (-\lambda)(-1-\lambda)-6 = \lambda^2+\lambda-6 = (\lambda+3)(\lambda-2)$, and so -3 and 2 are the eigenvalues. To find the eigenvectors, solve $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ and $\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$ to get $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. To find the eigenvalues of A^2, solve $det(A^2-\lambda I) = det(\begin{pmatrix} 7-\lambda & -3 \\ -2 & 6-\lambda \end{pmatrix} = (7-\lambda)(6-\lambda)-6 = \lambda^2-13\lambda+36 = (\lambda-9)(\lambda-4)$ so $\lambda = 9$ and $\lambda = 4$ work. The eigenvectors are the same, because $A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$.

2. Well, we have that $det(A-\lambda I) = det(A-\lambda I)^t = det(A^t-(\lambda I)^t) = det(A^t-\lambda I)$. Thus the roots of these two polynomials are equal. However, if $A = \begin{pmatrix} 1 & 1 \\ 0 & 7 \end{pmatrix}$, then A has $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as an eigenvalue, while $A^t = \begin{pmatrix} 1 & 0 \\ 1 & 7 \end{pmatrix}$ has $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and one can easily check that each of these is not an eigenvalue of the other.

3. (3,3 and 4 points)

a)Well, $A = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$ fits the bill. The eigenvalues are found by $det\begin{pmatrix} 1/2-\lambda & 1/2 \\ 1 & -\lambda \end{pmatrix} = (-\lambda)(1/2-\lambda)-1/2 = \lambda^2-\lambda/2-1/2 = (\lambda-1)(\lambda+1/2)$, so we get $\lambda_1 = 1$ and $\lambda_2 = -1/2$. To find the eigenvectors, we solve $\begin{pmatrix} -1/2 & 1/2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$, yielding $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 1/2 \\ 1 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$, yielding $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

b) Therefore, $S = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$, and $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}$. So $\Lambda^n = \begin{pmatrix} 1 & 0 \\ 0 & -(1/2)^n \end{pmatrix}$, and thus $lim_{n\to\infty}\Lambda^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and so $lim_{n\to\infty}A^n = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix}(-1/3) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix}(-1/3) = \begin{pmatrix} -2 & -1 \\ -2 & -1 \end{pmatrix}(-1/3)$.

c) $lim_{n\to\infty}A^n\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-1/3)\begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}$.

4. Well, $(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I) = (S\Lambda S^{-1}-\lambda_1 I)(S\Lambda S^{-1}-\lambda_2 I)\cdots(S\Lambda S^{-1}-\lambda_n I) = S(\Lambda-\lambda_1 I)\cdots(\Lambda-\lambda_n I)S^{-1}$. Now the matrix $\Lambda-\lambda_i I$ is diagonal with $\lambda_k-\lambda_i$ in the kth spot, and thus is 0 when $k=i$. Now one easily checks that the product is zero.

5. To find the eigenvalues of B, we compute $det(B-\lambda I) = det\begin{pmatrix} 3-\lambda & 2 \\ -5 & -3-\lambda \end{pmatrix} = -(3+\lambda)(3-\lambda)+10 = \lambda^2-9+10 = \lambda^2+1 = (\lambda+i)(\lambda-i)$. So the eigenvalues are $i,-i$, and thus B is diagonalizable: $B = S\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}S^{-1}$, so $B^4 = S\begin{pmatrix} i^4 & 0 \\ 0 & (-i)^4 \end{pmatrix}S^{-1} = SIS^{-1} = I$. For C, we consider $det(C-\lambda I) = det\begin{pmatrix} 5-\lambda & 7 \\ -3 & -4-\lambda \end{pmatrix} = -(4+\lambda)(5-\lambda)+21 = \lambda^2-\lambda+1$. I claim that $e^{\pi i/3}$ and $e^{-\pi i/3}$ are the roots (if you are unsure about this, use sin's and cos's). Given this, argue as above that $B^3 = -I$, using that $(e^{\pi i/3})^3 = e^{\pi i} = -1$.

6. (5 points each)

1. $det(A-\lambda I) = det\begin{pmatrix} .9-\lambda & .15 \\ .1 & .85-\lambda \end{pmatrix} = (.9-\lambda)(.85-\lambda)-.015 = \lambda^2-(1.75)\lambda+0.75 = (\lambda-1)(\lambda-.75)$. To find the eigenvector, we must solve $\begin{pmatrix} -0.1 & .15 \\ 0.1 & 0.15 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$, and arrive at $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

2. The other eigenvector is found by solving $\begin{pmatrix} 0.15 & 0.15 \\ 0.1 & 0.1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$, and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ works. So we see that $A = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix}(-1/5)$. So the limit in question is $\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix}(-1/5) = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$.

7. Starting with $\begin{pmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ & & \end{pmatrix}$, and using the rule that each column must add to one, we obtain $\begin{pmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ .2 & .3 & .5 \end{pmatrix}$. The steady state eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. This is so for any symmetric Markov matrix because all the rows must add to one also (since $A = A^t$), and so A acting on the vector consisting of 1's yields itself.

8. (5 points each)

1. a) $||\mathbf{v}|| = (\sum 1/2^n)^{1/2} = \sqrt{2}$.

b) $||\mathbf{v}|| = (\sum a^{2n})^{1/2} = (\sum (a^2)^n)^{1/2} = 1/(1-a^2)^{1/2}$.

c) $||\mathbf{v}||^2 = \int_0^{2\pi}(1+sin(x))^2dx = \int_0^{2\pi}(1+2sin(x)+sin^2(x))dx = 2\pi+0+\pi = 3\pi$, so $||\mathbf{v}|| = \sqrt{3\pi}$.

2. We recall that the fourier series for the "square wave" function sq was given in the text as $sq = (4/\pi)(sin(x) + sin(3x)/3 + sin(5x)/5 + ...)$. Now we are considering $f = 1/2 + (1/2)sq$ so its expansion must be $1/2 + (2/\pi)(sin(x) + sin(3x)/3 + sin(5x)/5 + ...)$. As for $f(x) = x$, an easy integration by parts reveals $a_0 = \pi$, while all other $a_k = 0$, and $b_k = -2/k$.

9. (3,3 and 4 points)

1. No. Choose $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $det(A - \lambda I) = \lambda^2$ whose only roots are zero, while A is not zero.

2. No. Same example as in 1.

3. Yes. Let $A^2 = 0$. Then certainly $det(A) = 0$ (A cannot be invertible. If $A = 0$ this is clearly so, if not, $A^{-1}A^2 = A$ but also $A^{-1}A^2 = A^{-1}0 = 0$, a contradiction). However, if $\lambda \neq 0$, then $(A - \lambda I)((1/\lambda)A + I) = 0 + A - A - \lambda I = -\lambda I$. So we see that $(A - \lambda I)^{-1} = (-1/\lambda)((1/\lambda)A + I) = A - (1/\lambda)I$ makes sense for all $\lambda \neq 0$. Thus $p(\lambda) = det(A - \lambda I)$ has only $\lambda = 0$ as a root.

10. (3,4 and 3 points)

1. No. Consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

2. Yes, A^{-1} also has only 1's as eigenvalues. Suppose that $A^{-1}\mathbf{y} = \lambda \mathbf{y}$. Then $A\mathbf{y} = \lambda^{-1}\mathbf{y}$. Similarly, $A\mathbf{y} = \lambda \mathbf{y}$ implies $A^{-1}\mathbf{y} = \lambda^{-1}\mathbf{y}$. Therefore, A and A^{-1} have the same eigenvectors, with inverse eigenvalues; so A^{-1} has only 1's as eigenvalues.

3. Consider $A = \begin{pmatrix} x & 1 \\ 0 & y \end{pmatrix}$.