SOLUTIONS TO PSET 5

1. Well, $AA^{-1} = I$ means: if $A_1, ..., A_n$ denote the rows of A, and $B_1, ..., B_n$ denote the columns of A^{-1}, then $A_i B_j = \delta_{ij}$ where the symbol δ_{ij} is equal to 1 if $i = j$ and 0 if $i \neq j$. But this says that B_1 is orthogonal to the space spanned by $\{A_2, ..., A_n\}$.

2. (5 points each)

a) $A^tA = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, while $A^t\mathbf{b} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. So our equation is $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \hat{x_1} \\ \hat{x_2} \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, yielding $\hat{x_2} = 3$ and $\hat{x_1} = -1$. So $\mathbf{p} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$, and $\mathbf{e} = \mathbf{b} - \mathbf{p} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.

b) $A^tA = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$, while $A^t\mathbf{b} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \end{pmatrix}$. So our equation is $\begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \hat{x_1} \\ \hat{x_2} \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \end{pmatrix}$, yielding $\hat{x_2} = 6$ and $\hat{x_1} = -2$. So $\mathbf{p} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$, and $\mathbf{e} = \mathbf{b} - \mathbf{p} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. (Note that this makes sense: $\mathbf{b}$ is already in the column space of A).

3. Well, $A(A^tA)^{-1}A^tA(A^tA)^{-1}A^t = A(A^tA)^{-1}A^t$ (just group the terms appropriately), so this says that $P^2 = P$. This makes sense because $P\mathbf{b}$ is already in the column space of A; so the projection of $P\mathbf{b}$ to the column space of A must be the $P\mathbf{b}$.

4. Well, we know that A^tA is invertible iff A has linearly independent columns (this is 4G in the book). Now, B has linearly independent rows iff B^t has linearly independent columns. Thus, the assumption implies that $(B^t)^tB^t = BB^t$ is invertible.

5. (5 point for each part).

1) We know (pg. 210, numbers (6) and (7)) that $A^tA = \begin{pmatrix} 4 & 8 \\ 8 & 26 \end{pmatrix}$ and $A^t\mathbf{b} = \begin{pmatrix} 36 \\ 112 \end{pmatrix}$. Solving $\begin{pmatrix} 4 & 8 \\ 8 & 26 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 36 \\ 112 \end{pmatrix}$ gives $C = 1$ and $D = 4$, yielding the line $1 + 4t$. The heights are given by $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 13 \\ 17 \end{pmatrix}$, and the errors by $\begin{pmatrix} 0-1 \\ 8-5 \\ 8-13 \\ 20-17 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -5 \\ 3 \end{pmatrix}$. The sum of the squares is 44 .

2) The four equations are $\begin{pmatrix}1 & 0\\ 1 & 1\\ 1 & 3\\ 1 & 4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 8\\ 8\\ 20\end{pmatrix}$, i.e., $x_1 = 0, x_1 + x_2 = 8, x_1 + 3x_2 = 8$, and $x_1 + 4x_2 = 20$ (looking at the first three equations tells you right away that the system is unsolvable). Changing the measurements to $\begin{pmatrix}1\\ 5\\ 13\\ 17\end{pmatrix}$ does yield a solution: $(1,4)$, as shown above.

6. Applying the method, we have the unsolvable system $\begin{pmatrix}1 & -1\\ 1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}C\\ D\end{pmatrix} = \begin{pmatrix}7\\ 7\\ 21\end{pmatrix}$.

So we multiply both sides by A^t to get $A^tA = \begin{pmatrix}1 & 1 & 1\\ -1 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\\ 1 & 2\end{pmatrix} = \begin{pmatrix}3 & 2\\ 2 & 6\end{pmatrix}$ and $A^t\mathbf{b} = \begin{pmatrix}1 & 1 & 1\\ -1 & 1 & 2\end{pmatrix}\begin{pmatrix}7\\ 7\\ 21\end{pmatrix} = \begin{pmatrix}35\\ 42\end{pmatrix}$, and we solve $\begin{pmatrix}3 & 2\\ 2 & 6\end{pmatrix}\begin{pmatrix}C\\ D\end{pmatrix} = \begin{pmatrix}35\\ 42\end{pmatrix}$ to get $C = 9$ and $D = 4$.

7. We know that Q is orthogonal iff $Q^tQ = I$. But, we have that if Q_1 and Q_2 are orthogonal, then $(Q_1Q_2)^t(Q_1Q_2) = Q_2^tQ_1^tQ_1Q_2 = Q_2^tQ_2 = I$.

8. (5 points each)

1) As $\mathbf{a} = \begin{pmatrix}1\\ 1\end{pmatrix}$, a vector $\mathbf{b}$ is orthogonal to $\mathbf{a}$ iff $\mathbf{b} = \begin{pmatrix}\lambda\\ -\lambda\end{pmatrix}$. But $\mathbf{b} - 2\mathbf{a} = \begin{pmatrix}2\\ -2\end{pmatrix}$.

2) $\mathbf{q}_1 = \mathbf{a}/||\mathbf{a}|| = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$, and $\mathbf{q_2} = \mathbf{B}/||\mathbf{B}|| = \begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}$. Finally $\begin{pmatrix}1 & 4\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}\sqrt{2} & ?\\ 0 & 2\sqrt{2}\end{pmatrix}$ implies that $? = 2\sqrt{2}$.

9. (5 points each)

a) The columns of Q are clearly mutally orthogonal, so we just need to get the norms to be 1. Since the norm of all these columns is clearly 2, we set $c = 1/2$.

b) $Q = 1/2\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\\ 1 & 1 & -1 & -1\end{pmatrix}$ will work. There are other choices; but the second column determines everything.