SOLUTIONS TO PSET 4

1. a) (5 points) $\begin{pmatrix}1&2&1\\2&6&3\\0&2&5\end{pmatrix} \to \begin{pmatrix}1&2&1\\0&2&1\\0&2&5\end{pmatrix} \to \begin{pmatrix}1&2&1\\0&2&1\\0&0&4\end{pmatrix}$ so this matrix is invertible. Therefore, the column space consists of all vectors in $\mathbb{R}^3$, and no nontrivial linear combo. of the rows can be zero.

b) (5 points) $\begin{pmatrix}1&1&1&b_1\\1&2&4&b_2\\2&4&8&b_3\end{pmatrix} \to \begin{pmatrix}1&1&1&b_1\\0&1&3&b_2-b_1\\0&2&6&b_3-2b_1\end{pmatrix} \to \begin{pmatrix}1&1&1&b_1\\0&1&3&b_2-b_1\\0&0&0&b_3-2b_2\end{pmatrix}$.
So the column space consists of $\{(b_1,b_2,b_3)|b_3-2b_2=0\}$. The last two rows are linearly dependant.

2. a) (5 points) $A = \begin{pmatrix}1&4&0\\2&11&5\\-1&2&10\end{pmatrix} \to \begin{pmatrix}1&4&0\\0&3&5\\0&6&10\end{pmatrix} \to \begin{pmatrix}1&4&0\\0&3&5\\0&0&0\end{pmatrix}$ therefore $2 = rk(A) = rk(A^t)$.

b) (5 points) $A = \begin{pmatrix}1&0&1\\1&1&2\\1&1&q\end{pmatrix} \to \begin{pmatrix}1&0&1\\0&1&1\\0&1&q-1\end{pmatrix} \to \begin{pmatrix}1&0&1\\0&1&1\\0&0&q-2\end{pmatrix}$. So the rank is 3 if $q \neq 2$, and 2 if $q = 2$.

3. a) (5 points) $\begin{pmatrix}2&4&4\\0&3&6\\0&0&0\end{pmatrix} \to \begin{pmatrix}1&2&2\\0&1&2\\0&0&0\end{pmatrix} \to \begin{pmatrix}1&0&-2\\0&1&2\\0&0&0\end{pmatrix}$.

b) (5 points) $\begin{pmatrix}2&4&4\\0&3&6\\0&0&5\end{pmatrix} \to \begin{pmatrix}1&2&2\\0&1&2\\0&0&1\end{pmatrix} \to \begin{pmatrix}1&1&0\\0&1&2\\0&0&1\end{pmatrix} \to \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

4. Well, U having n pivots means that, after rescaling, we can suppose all the diagonals are equal to 1. So the last column has $n-1$numbers which can be anything, and a 1 at the bottom. But this means that we can eliminate upward to kill all of those $n-1$ numbers, and get that that last column has $n-1$ zeros and a 1 at the bottom. Now proceed leftward.

5. Well, we first consider $\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$; this is in reduced form and the fact that the diagonals are all nonzero gives invertibility. So these three columns are linearly independant.

For the second part, just note that $4v_3 - v_2 - v_1 - v_4 = 0$.

6. Consider a linear combo. $c_1v_1 + c_2v_2 + c_3v_3 = 0$. This is the same as $c_1(w_2+w_3) + c_2(w_1+w_3) + c_3(w_1+w_2) = 0$, which is $(c_2+c_3)w_1 + (c_1+c_3)w_2 + (c_1+c_2)w_3 = 0$. But the linear independance of the w's now implies that $c_2+c_3 = c_1+c_3 = c_1+c_2 = 0$. The last two equations give $c_2 = c_3$, but then the first term gives $c_2 = 0$ and the conclusion follows.

7. Well, clearly $C(A) = \{(x_1,x_2,0)\}$ and $C(A^t) = \{(0,x_2,x_3)\}$. Further, since A is in reduced form we see that $N(A) = \{(x_1,0,0)\}$ and by inspection $N(A^t) = \{(0,0,x_3)\}$. Upon adding I, A becomes invertible, so $N(A) = N(A^t) = 0$ while $C(A) = C(A^t) = \mathbb{R}^3$.

8. (2.5 points each)

a) The column space is spanned by $\{v_i\mathbf{u}+z_i\mathbf{w}\}$. We note that this is always contained in the $\{\mathbf{u},\mathbf{w}\}$ plane.

b) The row space is spanned by $\{u_i\mathbf{v}^t+w_i\mathbf{z}^t\}$; note that this is contained in the $\{\mathbf{v}^t,\mathbf{u}^t\}$ plane.

c) If $\mathbf{v}=\mathbf{z}$ or if $\mathbf{u}=\mathbf{w}$.

d) $A=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$ which has rank 2.

9. $A=\begin{pmatrix}-1&1&0&0\\-1&0&1&0\\0&-1&1&0\\0&-1&0&1\\0&0&-1&1\end{pmatrix}\to\begin{pmatrix}-1&1&0&0\\0&-1&1&0\\0&-1&1&0\\0&-1&0&1\\0&0&-1&1\end{pmatrix}\to\begin{pmatrix}-1&1&0&0\\0&-1&1&0\\0&0&0&0\\0&0&-1&1\\0&0&-1&1\end{pmatrix}\to$
$\begin{pmatrix}-1&1&0&0\\0&-1&1&0\\0&0&-1&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$ (after a row permutation in the last step). So this is the incidence matrix for the graph which looks like $1\to 2\to 3\to 4$. A *tree* is a graph without loops, and which contains all four edges of the original graph. One ignores edge directions when counting trees; staring at the graph in question produces seven more.

10. We note that $(x_1-x_2)+(x_2-x_3)=x_1-x_3$. So; adding the first two equations and subtracting the third gives $0=1$; i.e., $y_1=y_2=1$, and $y_3=-1$.

11. Given that $S\subseteq V$, suppose that $\mathbf{w}\in V^{\perp}$. Then $\mathbf{v}\cdot\mathbf{w}=0$ for all $\mathbf{v}\in V$, so certainly $\mathbf{v}\cdot\mathbf{w}=0$ for all vectors $\mathbf{v}\in S$. But this says that $\mathbf{w}\in S^{\perp}$.