SOLUTIONS TO PSET 3

Problem 1. a) (5 points) Choose the lattice consisting of points (x,y) such that x and y are integers.

b) (5 points) Well, the fact that cx stays in the set says that the set is a union of lines. So take the x-axis union the y-axis.

Problem 2. (2.5 points each)

a) False. The compliment of a subspace is never a subspace, as it doesn't contain 0.

b) True. The assumption implies that every column is zero.

c) True. Obviously $C(2A) \subseteq C(A)$ as each column of $2A$ is obtained as a linear combo. of columns of A (namely multiplying the corresponding column by 2). But $C(A) \subseteq C(2A)$ as well, as we can also multiply by $1/2$.

d) False. Use $A = I$.

Problem 3. 1a) (2.5 points) we get $\begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix} \to$ $\begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ so the pivot variables are x_1 and x_3.

1b) (2.5 points) we get $\to \begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix}$; so x_1 and x_2 are the pivot variables.

2) (5 points) We easily arrive at $(-2,1,0,0,0)$, $(0,0,-2,1,0)$, and $(0,0,-3,0,1)$ for a), $(1,-1,1)$ for b).

Problem 4. Given the special solution $(12,0,0)$, we can find the general solution by adding solutions to the equation $x - 3y - z = 0$. Filling in $(1,0)$ for (y,z) gives $x = 3$ and filling in $(0,1)$ gives $x = 1$.

Problem 5. $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{pmatrix}$, $\begin{pmatrix} 2 & 6 & -3 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{pmatrix}$, and $\begin{pmatrix} a & b \\ c & bc/a \end{pmatrix}$ all have rank 1.

Problem 6. We know that $rank(I) = n$, so we see that $n \leq rank(A)$, and since A is $n \times n$, we have also $rank(A) \leq n$, so $rank(A) = n$. Thus A is invertible as required.

Problem 7. a) (5 points) The rows of the reduced echelon form are obtained by multiplying on the left by a sequence of triangular (and hence invertible) matrices. So it follows that the rowspace of a matrix and its echelon form are the same; and the same reasoning for the nullspace.

b) (5 points) The answer is "upper triangular", as indicated in part a).

Problem 8. We use elimination: $\begin{pmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 1 & 0 & b_3 - 4b_1 \end{pmatrix} \to$ $\begin{pmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{pmatrix}$. So the condition is $b_3 - 2b_1 - b_2 = 0$. Further, we see from the fact that x_1 and x_2 are pivot variables that there is a uniques special solution to $Ax = 0$, which is $(2,0,1)$.

Problem 9. The rank is the number of nonzero singular values.