SOLUTIONS TO PSET 10

Problem 1. The wavelet basis is $\mathbf{w}_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$, $\mathbf{w}_2 = \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$, $\mathbf{w}_3 = \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$, $\mathbf{w}_4 = \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$. If $\mathbf{e} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, then $\mathbf{e} = (1/4)(\mathbf{w}_1 + \mathbf{w}_2 + 2\mathbf{w}_3)$, and $\mathbf{v} = \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = \mathbf{w}_3 + \mathbf{w}_4$.

Problem 2. We have that $b_1\mathbf{v}_1 + ... + b_n\mathbf{v}_n = V\mathbf{b} = c_1\mathbf{w}_1 + ... + c_n\mathbf{w}_n = W\mathbf{c}$. Therefore $\mathbf{b} = V^{-1}W\mathbf{c}$, and so $V^{-1}W$ is M, the change of basis matrix.

Problem 3. Well, $W^* = (W^{-1})^t$, and this implies that $(W^*)^* = ((W^*)^{-1})^t = (((W^{-1})^t)^{-1})^t = W$ (use that inverse and transpose commute).

Problem 4. (5 points each)

1. $A = \begin{pmatrix}3 & 3\\-1 & 1\end{pmatrix}$. Therefore, $A^tA = \begin{pmatrix}3 & -1\\3 & 1\end{pmatrix}\begin{pmatrix}3 & 3\\-1 & 1\end{pmatrix} = \begin{pmatrix}10 & 8\\8 & 10\end{pmatrix}$. We find the eigenvalues by computing $det\begin{pmatrix}10-\lambda & 8\\8 & 10-\lambda\end{pmatrix} = (10-\lambda)^2 - 64 = 100 + \lambda^2 - 20\lambda - 64 = \lambda^2 - 20\lambda + 36 = (\lambda - 18)(\lambda - 2)$. So the singular values are $\sqrt{18}$ and $\sqrt{2}$. For the eigenvalue $\lambda = 18$, we find the unit eigenvector by solving $\begin{pmatrix}-8 & 8\\8 & -8\end{pmatrix}\begin{pmatrix}\mathbf{v}_1\\\mathbf{v}_2\end{pmatrix} = 0$, and so we get $\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$. For the eigenvalue $\lambda = 2$, we look at $\begin{pmatrix}8 & 8\\8 & 8\end{pmatrix}\begin{pmatrix}\mathbf{w}_1\\\mathbf{w}_2\end{pmatrix} = 0$, and we get $\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$.

2. With the same A, $AA^t = \begin{pmatrix}3 & 3\\-1 & 1\end{pmatrix}\begin{pmatrix}3 & -1\\3 & 1\end{pmatrix} = \begin{pmatrix}18 & 0\\0 & 2\end{pmatrix}$, so we see the same eigenvalues, and arrive at eigenvectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. From this we obtain the SVD for A which reads $A = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{18} & 0\\0 & \sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\-1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$.

Problem 5. Suppose A is $n \times n$. If $det(A) = 0$, then it must be that $rk(A) < n$. But we have that $rk(A^+) = rk(A)$ (this follows from the defintion of $A^+ = V\Sigma^+U^T$; the rank of A is the number of nonzero singular values, which is also the number of nonzero entries of Σ^+). Thus A^+ doesn't have full rank, and so it isn't invertible, and so $det(A^+) = 0$.

Problem 6. Let $\hat{x}$ be any solution to $A^TA\hat{x} = A^Tb$. Then as $A^TAx^+ = A^Tb$, we see that $A^TA(\hat{x} - x^+) = A^Tb - A^Tb = 0$. Thus $\hat{x} - x^+$ is in $N(A^TA) = N(A)$. Now, as $N(A)^\perp = Row(A)$, the equality $||\hat{x}||^2 = ||x^+||^2 + ||\hat{x} - x^+||^2$ will follow if we can show that $x^+ = A^+b$ is in $Row(A)$. But recall the defintion of A^+ : we had that $A^+u_i = \sigma_i^{-1}v_i$ (for $1 \le i \le r$), and $A^+u_i = 0$ for $i > r$. Thus the image A^+ is exactly $span\{v_1, ..., v_r\} = Row(A)$.

Problem 7. Well, AA^+ is the projection onto the column space of A, while A^+A is the projection onto the rowspace. Therefore, if $b = p + e$ is the decomposition of b into its column space and left nullspace part, then $AA^+p = p$ while $AA^+e = 0$. Similarly, $A^+Ax_r = x_r$, while $A^+Ax_n = 0$.