SOLUTIONS TO THE PRACTICE SET

Problem 1. We know (section 6.4) that Λ is a real diagonal matrix. Thus $\Lambda^H = \Lambda$, and $A^H = (U^H \Lambda U)^H = U^H \Lambda^H (U^H)^H = A$. So A is hermetian and hence has real eigenvalues (section 10.2). Further, since $U^H = U^{-1}$, A must have the same eigenvalues as Λ, namely the diagonal elements of Λ (so A doesn't have to be postive definite). However, if the eigenvectors of Λ are given by $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$, then the eigenvectors of A are given by $\{U^{-1}\mathbf{x}_1, ..., U^{-1}\mathbf{x}_n\}$. So, the allthough the $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ are real, the $\{U^{-1}\mathbf{x}_1, ..., U^{-1}\mathbf{x}_n\}$ might not be (as U is a complex matrix). The diagonal elements of A are real as A is hermetian, but the off-diagonal elements might not be.

Problem 2. If $\mathbf{x}$ is an eigenvector, then so is $c\mathbf{x}$ for all nonzero c. If $\mathbf{x}$ has unit length, then $||c\mathbf{x}|| = |c| \cdot ||\mathbf{x}|| = |c|$, so $c\mathbf{x}$ has unit length iff c does.

Problem 3. If A is $n \times n$ and Markov, then each entry is nonnegative and ≤ 1, so $0 \leq Tr(A) = a_{11} + a_{22} + ... + a_{nn} \leq 1 + 1 + ... + 1 = n$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $det(A) = ad - bc = a(1-b) - b(1-a) = a - ab - b + ab = a - b$. As both a and b fall between 0 and 1, this gives the inequality $-1 \leq det(A) \leq 1$.

Problem 4. The sum of the eigenvalues is the trace. Thus, if A has 1 as a triple eigenvalue, then $Tr(A) = 3$. But as each entry in A is postive and ≤ 1, this must mean that (as $3 = Tr(A) = a_{11} + a_{22} + a_{33}$), we have $a_{ii} = 1$ and thus, as each column sums to 1, we have that $A = I$.

Problem 5. This is impossible. For any M and A, $Tr(MAM^{-1}) = Tr((MA)M^{-1}) = Tr(M^{-1}(MA)) = Tr(A)$. As the traces of the two matrices in question are different, we see that it can't happen.

Problem 6. If A is diagonalizable with eigenvalues 0 and 1, we have that $A = S\Lambda S^{-1}$, so $A^2 = S\Lambda^2 S^{-1}$. But $0^2 = 0$ and $1^2 = 1$, so $\Lambda^2 = \Lambda$, so $A^2 = A$.

Problem 7. Yes. In fact, A must be diagonalizable, since the fact that it has two distinct eigenvalues means that it has two distinct eigenvectors, and it is 2×2. If A is 3×3, then the answer is no. If we let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$, then we see that $A^4 = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which is not diagonalizable, as 1 is its only eigenvalue, but $A - I = \begin{pmatrix} 0 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and the nullspace of this matrix is only 2 dimensional.