18.06 Problem Set 9

Due Wednesday, 21 November 2007 at 4 pm in 2-106.

Problem 1: When $A = S\Lambda S^{-1}$ is a real-symmetric (or Hermitian) matrix, its eigenvectors can be chosen orthonormal and hence $S = Q$ is orthogonal (or unitary). Thus, $A = Q\Lambda Q^T$, which is called the *spectral decomposition* of A.

Find the spectral decomposition for $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, and check by explicit multiplication that $A = Q\Lambda Q^T$. Hence, find A^{-3} and $\cos(A\pi/3)$.

Problem 2: Suppose A is any $n \times n$ real matrix.
(1) If $\lambda \in \mathbb{C}$ is an eigenvalue of A, show that its complex conjugate $\bar{\lambda}$ is also an eigenvalue of A. (Hint: take the complex-conjugate of the eigen-equation.)
(2) Show that if n is odd, then A has at least one real eigenvalue. (Hint: think about the characteristic polynomial.)

Problem 3: In class, we showed that a Hermitian matrix (or its special case of a real-symmetric matrix) has real eigenvalues and that eigenvectors for distinct eigenvalues are always orthogonal. Now, we want to do a similar analysis of *unitary* matrices $Q^H = Q^{-1}$ (including the special case of real orthogonal matrices).
(1) The eigenvalues λ of Q are *not* real in general—rather, they always satisfy $|\lambda|^2 = \bar{\lambda}\lambda = 1$. Prove this. (Hint: start with $Q\mathbf{x} = \lambda\mathbf{x}$ and consider the dot product $(Q\mathbf{x}) \cdot (Q\mathbf{x}) = \mathbf{x}^H Q^H Q\mathbf{x}$.)
(2) Prove that eigenvectors with distinct eigenvalues are orthogonal. (Hint: consider $(Q\mathbf{x}) \cdot (Q\mathbf{y})$ for two eigenvectors $\mathbf{x}$ and $\mathbf{y}$ with different eigenvalues. You will also find useful the fact that, since $|\lambda|^2 = 1 = \bar{\lambda}\lambda$, then $\bar{\lambda} = 1/\lambda$.)
(3) Give examples of 1×1 and 2×2 unitary matrices and show that their eigensolutions have the above properties. (Give both a purely real and a complex example for each size. Don't pick a diagonal 2×2 matrix.)

Problem 4: Consider the vector space of real twice-differentiable functions $f(x)$ defined for $x \in [0, 1]$ with $f(0) = f(1) = 0$, and the dot product $f \cdot g = \int_0^1 f(x)g(x)dx$. Use the linear operator A defined by

$$Af = -\frac{d}{dx}\left[w(x)\frac{df}{dx}\right],$$

where $w(x) > 0$ is some positive differentiable function.
(1) Show that A is Hermitian [use integration by parts to show $f \cdot (Ag) = (Af) \cdot g$] and positive-definite [by showing $f \cdot (Af) > 0$ for $f \neq 0$]. What can you conclude about the eigenvalues and eigenfunctions (even though you can't calculate them explicitly)?

(2) To solve for the eigenfunctions of A for most functions $w(x)$, we must do so numerically. The simplest approach is to replace the derivatives d/dx by approximate differences: $f'(x) \approx [f(x + \Delta x) - f(x - \Delta x)]/2\Delta x$ for some small Δx. In this way, we construct a finite $n \times n$ matrix A (for $n \approx 1/\Delta x$). This is done by the following Matlab code, given a number of points $n > 1$ and a function `w(x)`:

```
dx = 1 / (n - 1);
x = linspace(0,1,n);
A = diag(w(x+dx/2)+w(x-dx/2)) - diag(w(x(1:n-1)+dx/2), 1)
    - diag(w(x(2:n)-dx/2), -1);
A = A / (dx^2);
```

Now, set $n = 100$ and $w(x) = 1$:

```
n = 100
w = @(x) ones(size(x))
```

and then type in the Matlab code above to initialize A. In this case, the problem $-f'' = \lambda f$ is analytically solvable for the eigenfunctions $f_k(x) = \sin(k\pi x)$ and eigenvalues $\lambda_k = (k\pi)^2$. Run `[S,D] = eig(A)` to find the eigenvectors (columns of S) and eigenvalues (diagonal of D) of your 100×100 approximate matrix A from above, and compare the lowest 3 eigenvalues and the corresponding eigenvectors to the analytical solutions for $k = 1, 2, 3$. That is, do:

```
[S,D] = eig(A);
plot(x, S(:,1:3), 'ro', x, [sin(pi*x);sin(2*pi*x);sin(3*pi*x)]*sqrt(2*dx), 'k-')
```

to plot the numerical eigenfunctions (red dots) and the analytical eigenfunctions (black lines). (The `sqrt(2*dx)` is there to make the normalizations the same. You might need to flip some of the signs to make the lines match up.) You can compare the eigenvalues (find their ratios) by running:

```
lambda = diag(D);
lambda(1:3) ./ ([1:3]'*pi).^2
```

(3) Repeat the above process for $n = 500$ and show that the first three eigenvalues and eigenfunctions come closer to the analytical solutions as n increases.

(4) Try it for a different $w(x)$ function, for example $w(x) = \cos(x)$ (which is positive for $x \in [0, 1]$), and $n = 100$:

```
w = @(x) cos(x)
n = 100
```

After you construct A with this new $w(x)$ using the commands above, look at the upper-left 10×10 corner to verify that it is symmetric: type `A(1:10,1:10)`. Check that the eigenvalues satisfy your conditions from (1) by using the following commands to find the maximum imaginary part and the minimum real part of all the λ's, respectively:

```
[S,D] = eig(A);
lambda = diag(D);
max(abs(imag(lambda)))
min(real(lambda))
```

Plot the eigenvectors for the lowest three eigenvalues, as above, and compare them to the analytical solutions for the $w(x) = 1$ case. You will need to make sure that the eigenvalues are sorted in increasing order, which can be done with the `sort` command:

```
[lambda,order] = sort(diag(D));
Q = S(:,order);
plot(x, Q(:,1:3), 'r.-', x, [sin(pi*x);sin(2*pi*x);sin(3*pi*x)]*sqrt(2*dx), 'k-')
```

(Again, you may want to flip some of the signs to make the comparison easier.) Verify that the first three eigenvectors are still orthogonal (even though they are no longer simply sine functions) by computing the dot products:

```
Q(:,1)' * Q(:,2)
Q(:,1)' * Q(:,3)
Q(:,2)' * Q(:,3)
```

The numbers should be almost zero, up to roundoff error (14–16 decimal places).

Problem 5: True or False? Give reasons.
(1) If A is real symmetric matrix, then any two linearly independent eigenvectors of A are perpendicular.
(2) Any $n \times n$ complex matrix with n real eigenvalues and n orthonormal eigenvectors is a Hermitian matrix.
(3) If all entries of A are positive, then A is positive-definite matrix.
(4) If A, B are positive definite matrices, then $A + B$ is positive matrix.
(5) If A is positive-definite matrix, then A^{-1} is also a positive-definite matrix.

Problem 6: Find all 2×2 real matrices that are both symmetric and orthogonal.

Problem 7: For $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_2x_3)$, find a 3×3 symmetric matrix A such that $f = \mathbf{x}^T A\mathbf{x}$, and check whether A is positive definite (hint: the easiest way is to use the pivots).