18.06 Problem Set 9 - Solutions

Due Wednesday, 21 November 2007 at 4 pm in 2-106.

Problem 1: (15) When $A = S\Lambda S^{-1}$ is a real-symmetric (or Hermitian) matrix, its eigenvectors can be chosen orthonormal and hence $S = Q$ is orthogonal (or unitary). Thus, $A = Q\Lambda Q^T$, which is called the *spectral decomposition* of A.

Find the spectral decomposition for $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, and check by explicit multiplication that $A = Q\Lambda Q^T$. Hence, find A^{-3} and $\cos(A\pi/3)$.

$\boxed{\text{Solution}}$ The characteristic equation for A is $\lambda^2 - 6\lambda + 5 = 0$. Thus the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 5$. For $\lambda_1 = 1$, the eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. For $\lambda_2 = 5$, the eigenvector is $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus the spectral decomposition of A is

$$A = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}.$$

We check the above decomposition by explicit multiplication:

$$\begin{aligned} \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} &= \begin{pmatrix} \sqrt{2}/2 & 5\sqrt{2}/2 \\ -\sqrt{2}/2 & 5\sqrt{2}/2 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}. \end{aligned}$$

Now

$$\begin{aligned} A^{-3} &= \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5^{-3} \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \\ &= \begin{pmatrix} (1+5^{-3})/2 & (-1+5^{-3})/2 \\ (-1+5^{-3})/2 & (1+5^{-3})/2 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \cos(A\pi/3) &= \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} \cos(\pi/3) & 0 \\ 0 & \cos(5\pi/3) \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}. \end{aligned}$$

Problem 2: (10=5+5) Suppose A is any $n \times n$ real matrix.

(1) If $\lambda \in \mathbb{C}$ is an eigenvalue of A, show that its complex conjugate $\bar{\lambda}$ is also an eigenvalue of A. (Hint: take the complex-conjugate of the eigen-equation.)

$\boxed{\text{Solution}}$ Let $p(x)$ be the characteristic polynomial for A. Then $p(\lambda) = 0$. Take conjugate, we get $\overline{p(\lambda)} = \overline{0}$. Since A is a real matrix, p is a polynomial of real coefficient, which implies have $\overline{p(x)} = p(\bar{x})$ for all x. Thus $p(\bar{\lambda}) = 0$, i.e. , $\bar{\lambda}$ is an eigenvalue of A.

Another proof: Suppose $A\mathbf{x} = \lambda\mathbf{x}$, take conjugate, we get $A\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}$, so $\bar{\lambda}$ is an eigenvalue with eigenvector $\bar{\mathbf{x}}$.

(2) Show that if n is odd, then A has at least one real eigenvalue. (Hint: think about the characteristic polynomial.)

$\boxed{\text{Solution}}$ We have seen above that if λ is an eigenvalue of A, then $\bar{\lambda}$ is also an eigenvalue of A. In fact, we can say more: if $A\mathbf{v} = \lambda\mathbf{v}$, then by taking conjugate we have $A\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$, where we used the property that A is a real matrix. This shows that if $\mathbf{v}$ is an eigenvector of A corresponding to eigenvalue λ, then $\bar{\mathbf{v}}$ is an eigenvector of A corresponding to eigenvalue $\bar{\lambda}$. So all complex eigenvalues of A can be paired (counting multiplicity) to be conjugated pairs $(\lambda, \bar{\lambda})$. This implies that A has even number of complex eigenvalues. However, since n is odd, A has odd ($=n$) number of eigenvalues. Thus A has at least one real eigenvalue.

Another proof: The leading term of the characteristic polynomial $p(x)$ is λ^n. When n is odd, $p(x)$ will tend to $\pm\infty$ when x tends to $\pm\infty$. So $p(x)$ must has at least one real root.

Problem 3: (20=6+6+8) In class, we showed that a Hermitian matrix (or its special case of a real-symmetric matrix) has real eigenvalues and that eigenvectors for distinct eigenvalues are always orthogonal. Now, we want to do a similar analysis of *unitary* matrices $Q^H = Q^{-1}$ (including the special case of real orthogonal matrices).

(1) The eigenvalues λ of Q are *not* real in general—rather, they always satisfy $|\lambda|^2 = \bar{\lambda}\lambda = 1$. Prove this. (Hint: start with $Q\mathbf{x} = \lambda\mathbf{x}$ and consider the dot product $(Q\mathbf{x}) \cdot (Q\mathbf{x}) = \mathbf{x}^H Q^H Q\mathbf{x}$.)

$\boxed{\text{Solution}}$ Suppose λ is an eigenvalue of Q and $\mathbf{x}$ is an eigenvector of λ. Then one has

$$(Q\mathbf{x}) \cdot (Q\mathbf{x}) = (\lambda\mathbf{x}) \cdot (\lambda\mathbf{x}) = |\lambda|^2 \mathbf{x} \cdot \mathbf{x}.$$

On the other hand, since Q is unitary, we have

$$(Q\mathbf{x}) \cdot (Q\mathbf{x}) = \mathbf{x}^H Q^H Q\mathbf{x} = \mathbf{x}^H \mathbf{x} = \mathbf{x} \cdot \mathbf{x}.$$

Compare the above two equations, we get $|\lambda|^2 \mathbf{x} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{x}$. Since $\mathbf{x}$ is an eigenvector, we have $\mathbf{x} \neq 0$, thus $\mathbf{x} \cdot \mathbf{x} \neq 0$. So we must have $|\lambda|^2 = 1$.

(2) Prove that eigenvectors with distinct eigenvalues are orthogonal. (Hint: consider $(Q\mathbf{x}) \cdot (Q\mathbf{y})$ for two eigenvectors $\mathbf{x}$ and $\mathbf{y}$ with different eigenvalues. You will also find useful the fact that, since $|\lambda|^2 = 1 = \bar{\lambda}\lambda$, then $\bar{\lambda} = 1/\lambda$.)

$\boxed{\text{Solution}}$ Suppose λ_1 and λ_2 are distinct eigenvectors of A, with corresponding eigenvectors $\mathbf{x}$ and $\mathbf{y}$ respectively. Then as above, we have

$$(Q\mathbf{x}) \cdot (Q\mathbf{y}) = (\lambda_1 \mathbf{x}) \cdot (\lambda_2 \mathbf{y}) = \bar{\lambda}_1 \lambda_2 \mathbf{x} \cdot \mathbf{y}$$

and

$$(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x}^H Q^H Q \mathbf{y} = \mathbf{x}^H \mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

Compare the above two equations, we have

$$\bar{\lambda}_1 \lambda_2 \mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

From part (1) we have already seen that $|\lambda_1|^2 = \bar{\lambda}_1 \lambda_1 = 1$, thus $\bar{\lambda}_1 = 1/\lambda_1$. This implies $\bar{\lambda}_1 \lambda_2 = \lambda_2/\lambda_1 \neq 1$, since λ_1 are λ_2 are distinct. It follows that $\mathbf{x} \cdot \mathbf{y} = 0$, i.e. the eigenvectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

(3) Give examples of 1×1 and 2×2 unitary matrices and show that their eigensolutions have the above properties. (Give both a purely real and a complex example for each size. Don't pick a diagonal 2×2 matrix.)

$\boxed{\text{Solution}}$ For $n = 1$, A only has one entry which is its eigenvalue. Thus the only real examples are (1) and (-1), and complex examples are $(\cos\theta + i \sin\theta)$ for any θ.

For $n = 2$. The real example is given by any 2×2 orthogonal matrix. For example, we can take the matrix appeared in problem 1: $\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$. The eigenvalues of this matrix are $\lambda = \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i$. We have $|\lambda|^2 = 1$ for both of them. The corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}i/2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}i/2 \end{pmatrix}$. They are orthogonal: $\mathbf{v}_1 \cdot \mathbf{v}_2 = (\sqrt{2}/2)(\sqrt{2}/2) + (-\sqrt{2}i/2)(-\sqrt{2}i/2) = 0$.

To produce an complex example, we need to find two orthogonal complex vectors. For example, we may take the matrix $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. The eigenvalues are ± 1, both satisfies $|\lambda|^2 = 1$. The eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$. They are orthogonal since we have $\mathbf{v}_1 \cdot \mathbf{v}_2 = \bar{i} \times (-i) + \bar{1} \times 1 = -1 + 1 = 0$.

Problem 4: (20=8+4+4+4) Consider the vector space of real twice-differentiable functions $f(x)$ defined for $x \in [0, 1]$ with $f(0) = f(1) = 0$, and the dot product $f \cdot g = \int_0^1 f(x)g(x)dx$. Use the linear operator A defined by

$$Af = -\frac{d}{dx}\left[w(x)\frac{df}{dx}\right],$$

where $w(x) > 0$ is some positive differentiable function.

(1) Show that A is Hermitian [use integration by parts to show $f \cdot (Ag) = (Af) \cdot g$] and positive-definite [by showing $f \cdot (Af) > 0$ for $f \neq 0$]. What can you conclude about the eigenvalues and eigenfunctions (even though you can't calculate them explicitly)?

$\boxed{\text{Solution}}$ To show that A is Hermitian, we only need to show that $f \cdot (Ag) = (Af) \cdot g$ for all f and g in this vector space. We compute

$$\begin{aligned}
f \cdot (Ag) &= \int_0^1 f(x)(Ag)(x)dx \\
&= -\int_0^1 f(x)\frac{d}{dx}\left[w(x)\frac{dg}{dx}\right]\, dx \\
&= -\, f(x)w(x)\frac{dg}{dx}\Big|_0^1 + \int_0^1 w(x)\frac{dg}{dx}\frac{df}{dx}dx \\
&= \int_0^1 w(x)\frac{dg}{dx}\frac{df}{dx}dx.
\end{aligned}$$

By symmetry, we have

$$(Af) \cdot g = g \cdot (Af) = \int_0^1 w(x)\frac{df}{dx}\frac{dg}{dx}dx.$$

Thus $f \cdot (Ag) = (Af) \cdot g$, i.e. A is Hermitian.

Now suppose $f \neq 0$. Since $f(0) = f(1) = 0$, we see that f is not constant, i.e. $\frac{df}{dx} \neq 0$. Take $g = f$ in the above computation, we get

$$f \cdot (Af) = \int_0^1 w(x)\frac{df}{dx}\frac{df}{dx}dx > 0$$

since $w(x) > 0$. So A is positive definite.

So the eigenvalues of A are all positive real numbers, and eigenvectors corresponding to different eigenvalues are orthogonal to each other.

(2) To solve for the eigenfunctions of A for most functions $w(x)$, we must do so numerically. The simplest approach is to replace the derivatives d/dx by approximate differences: $f'(x) \approx [f(x+\Delta x) - f(x-\Delta x)]/2\Delta x$ for some small Δx. In this way, we construct a finite $n \times n$ matrix A (for $n \approx 1/\Delta x$). This is done by the following Matlab code, given a number of points $n > 1$ and a function `w(x)`:

```
dx = 1 / (n - 1);
x = linspace(0,1,n);
A = diag(w(x+dx/2)+w(x-dx/2)) - diag(w(x(1:n-1)+dx/2), 1)
    - diag(w(x(2:n)-dx/2), -1);
A = A / (dx^2);
```

Now, set $n = 100$ and $w(x) = 1$:

```
n = 100
w = @(x) ones(size(x))
```

and then type in the Matlab code above to initialize A. In this case, the problem $-f'' = \lambda f$ is analytically solvable for the eigenfunctions $f_k(x) = \sin(k\pi x)$ and eigenvalues $\lambda_k = (k\pi)^2$. Run `[S,D] = eig(A)` to find the eigenvectors (columns of S) and eigenvalues (diagonal of D) of your 100×100 approximate matrix A from above, and compare the lowest 3 eigenvalues and the corresponding eigenvectors to the analytical solutions for $k = 1, 2, 3$. That is, do:

```
[S,D] = eig(A);
plot(x, S(:,1:3), 'ro', x, [sin(pi*x);sin(2*pi*x);sin(3*pi*x)]*sqrt(2*dx), 'k-')
```

to plot the numerical eigenfunctions (red dots) and the analytical eigenfunctions (black lines). (The `sqrt(2*dx)` is there to make the normalizations the same. You might need to flip some of the signs to make the lines match up.) You can compare the eigenvalues (find their ratios) by running:

```
lambda = diag(D);
lambda(1:3) ./ ([1:3]'*pi).^2
```

$\boxed{\text{Solution}}$ The commands are

```
>> n=100;w=@(x) ones(size(x));
>> dx=1/(n-1);
>> x=linspace(0,1,n);
>> A=diag(w(x+dx/2)+w(x-dx/2))-diag(w(x(1:n-1)+dx/2),1)-diag(w(x(2:n)-dx/2),-1);
>> A=A/(dx^2);
>> [S,D]=eig(A);
>> plot(x,S(:,1:3),'ro',x,[-sin(pi*x);-sin(2*pi*x);-sin(3*pi*x)]*sqrt(2*dx),'k-')
>> lambda=diag(D);
>> lambda(1:3)./([1:3]'*pi).^2

ans =

    0.9607
    0.9605
    0.9601
```

The graph is

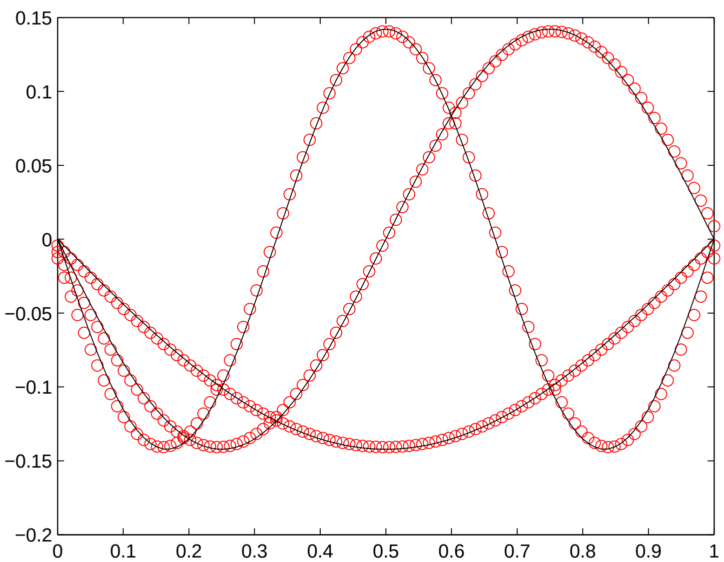

Figure 1: Eigenfunctions

(3) Repeat the above process for $n = 500$ and show that the first three eigenvalues and eigenfunctions come closer to the analytical solutions as n increases.

Solution The commands

```
>> n=500;w=@(x) ones(size(x));
>> dx=1/(n-1);
>> x=linspace(0,1,n);
>> A=diag(w(x+dx/2)+w(x-dx/2))-diag(w(x(1:n-1)+dx/2),1)-diag(w(x(2:n)-dx/2),-1);
>> A=A/(dx^2);
>> [S,D]=eig(A);
>> plot(x,S(:,1:3),'ro',x,[-sin(pi*x);-sin(2*pi*x);-sin(3*pi*x)]*sqrt(2*dx),'k-')
>> lambda=diag(D);
>> lambda(1:3)./([1:3]'*pi).^2
ans =
    0.9920
    0.9920
    0.9920
```

The graph is

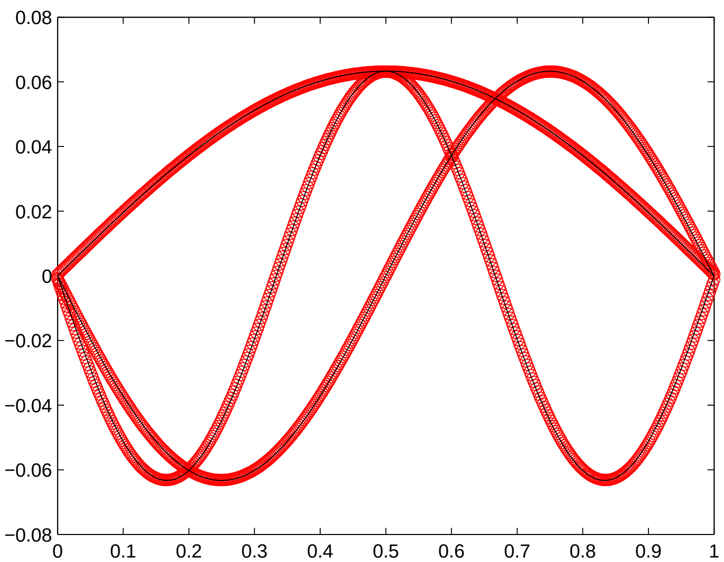

Figure 2: Eigenfunctions

(4) Try it for a different $w(x)$ function, for example $w(x) = \cos(x)$ (which is positive for $x \in [0, 1]$), and $n = 100$:

```
w = @(x) cos(x)
n = 100
```

After you construct A with this new $w(x)$ using the commands above, look at the upper-left 10×10 corner to verify that it is symmetric: type `A(1:10,1:10)`. Check that the eigenvalues satisfy your conditions from (1) by using the following commands to find the maximum imaginary part and the minimum real part of all the λ's, respectively:

```
[S,D] = eig(A);
lambda = diag(D);
max(abs(imag(lambda)))
min(real(lambda))
```

Plot the eigenvectors for the lowest three eigenvalues, as above, and compare them to the analytical solutions for the $w(x) = 1$ case. You will need to make sure that the eigenvalues are sorted in increasing order, which can be done with the `sort` command:

```
[lambda,order] = sort(diag(D));
Q = S(:,order);
plot(x, Q(:,1:3), 'r.-', x, [sin(pi*x);sin(2*pi*x);sin(3*pi*x)]*sqrt(2*dx), 'k-')
```

(Again, you may want to flip some of the signs to make the comparison easier.) Verify that the first three eigenvectors are still orthogonal (even though they are no longer simply sine functions) by computing the dot products:

```
Q(:,1)' * Q(:,2)
Q(:,1)' * Q(:,3)
Q(:,2)' * Q(:,3)
```

The numbers should be almost zero, up to roundoff error (14–16 decimal places).

$\boxed{\text{Solution}}$ The commands

```
>> n=100;w=@(x) cos(x);
>> dx=1/(n-1);
>> x=linspace(0,1,n);
>> A=diag(w(x+dx/2)+w(x-dx/2))-diag(w(x(1:n-1)+dx/2),1)-diag(w(x(2:n)-dx/2),-1);
>> A=A/(dx^2);
```

```
>> A(1:10,1:10)

ans =

   1.0e+04 *

 1.9602 -0.9801       0       0       0       0       0       0       0       0
-0.9801  1.9601 -0.9800       0       0       0       0       0       0       0
      0 -0.9800  1.9598 -0.9798       0       0       0       0       0       0
      0       0 -0.9798  1.9593 -0.9795       0       0       0       0       0
      0       0       0 -0.9795  1.9586 -0.9791       0       0       0       0
      0       0       0       0 -0.9791  1.9577 -0.9786       0       0       0
      0       0       0       0       0 -0.9786  1.9566 -0.9780       0       0
      0       0       0       0       0       0 -0.9780  1.9553 -0.9773       0
      0       0       0       0       0       0       0 -0.9773  1.9538 -0.9765
      0       0       0       0       0       0       0       0 -0.9765  1.9521

>> [S,D]=eig(A);
>> lambda=diag(D);
>> max(abs(imag(lambda)))

ans =

     0

>> min(real(lambda))

ans =

    7.5419

>> [lambda,order]=sort(diag(D));
>> Q=S(:,order);
>> plot(x,Q(:,1:3),'r.-',x,[-sin(pi*x);sin(2*pi*x);sin(3*pi*x)]*sqrt(2*dx),'k-')
>> Q(:,1)'*Q(:,2)

ans =

   1.0214e-14
```

```
>> Q(:,1)'*Q(:,3)

>> ans =

  -8.7823e-15

>> Q(:,2)'*Q(:,3)

ans =

  -9.8064e-14
```

The graph

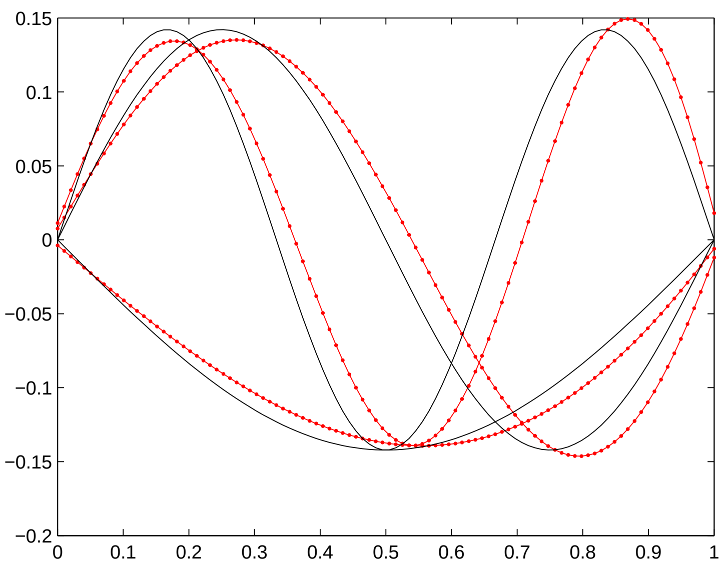

Figure 3: Eigenfunctions

Problem 5: (15=3+3+3+3+3) True or False? Give reasons.

(1) If A is real symmetric matrix, then any two linearly independent eigenvectors of A are perpendicular.

$\boxed{\text{Solution}}$ False.

For a symmetric or Hermitian matrix, eigenvectors corresponds to different eigenvalues are orthogonal. But independent eigenvectors corresponds to the same eigenvalue may not be perpendicular.

Counterexample: Take A to be the identity matrix. Then any vector is eigenvector. Two different vector don't have to be perpendicular.

(2) Any $n \times n$ complex matrix with n real eigenvalues and n orthonormal eigenvectors is a Hermitian matrix.

$\boxed{\text{Solution}}$ True.

Let Λ be the diagonal matrix with diagonal entries the eigenvalues, and S be the matrix of corresponding eigenvectors. Then $A = S\Lambda S^{-1}$. Since S consists of orthonormal vectors, it is unitary. Thus $A = S\Lambda S^H$. Since $\Lambda^H = \Lambda$, we see $A^H = S\Lambda^H S^H = S\Lambda S^H = A$. So A is Hermitian.

(3) If all entries of A are positive, then A is positive-definite matrix.

$\boxed{\text{Solution}}$ False.

Counterexample: Take $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Then $\det(A) = -3 < 0$, thus A must have negative eigenvalue, and thus A is not positive-definite.

(4) If A, B are positive definite matrices, then $A + B$ is positive matrix.

$\boxed{\text{Solution}}$ True.

Suppose A, B are positive definite, then for any $\mathbf{x}$, $\mathbf{x}^H A\mathbf{x} > 0$ and $\mathbf{x}^H B\mathbf{x} > 0$. Thus $\mathbf{x}^H(A + B)\mathbf{x} > 0$. In other words, $A + B$ is positive definite.

(To graders: If a student thinks that this "positive matrix" means a matrix whose entries are positive, answers "False" and gives an correct counterexample, please give full score.)

(5) If A is positive-definite matrix, then A^{-1} is also a positive-definite matrix.

$\boxed{\text{Solution}}$ True.

Since A is positive-definite, its eigenvalues $\lambda_1, \cdots, \lambda_n$ are all positive. Thus the eigenvalues of A^{-1}, $\lambda_1^{-1}, \cdots, \lambda_n^{-1}$, are all positive. So A^{-1} is positive-definite.

Problem 6: (10) Find all 2×2 real matrices that are both symmetric and orthogonal.

$\boxed{\text{Solution}}$ Any symmetric 2×2 matrix is of the form $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Since it is also orthogonal, two column vectors are perpendicular. So $ab + bc = 0$. In other words, $b = 0$ or $a + c = 0$.

If $b = 0$, then the matrix has the form $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. It is orthogonal if and only if $a = \pm 1$ and $c = \pm 1$, i.e. $A = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.

If $a + c = 0$, then the matrix has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. It is orthogonal if and only if $a^2 + b^2 = 1$, i.e. $a = \cos\theta$ and $b = \sin\theta$. So the matrix is $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ for some θ.

Problem 7: (10) For $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_2x_3)$, find a 3×3 symmetric matrix A such that $f = \mathbf{x}^T A\mathbf{x}$, and check whether A is positive definite (hint: the easiest way is to use the pivots).

$\boxed{\text{Solution}}$ The matrix A is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

To show that this is positive definite, we only need to find all the pivots:

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{pmatrix}.$$

The pivots are all positive, so A is positive definite.