18.06 Problem Set 8

Due Wednesday, 14 November 2007 at 4 pm in 2-106.

Problem 1: Consider the matrix $A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$.
(a) Check that A is a positive Markov matrix, and find its steady state.
(b) Factor A into $S\Lambda S^{-1}$.
(c) Explain why A^k approaches $A^\infty = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$ in two ways, using results in (a) and (b) respectively.
(d) Find all Markov matrices with steady state $(0.6, 0.4)^T$.

Problem 2: Suppose A is a Markov matrix, and let $\mathbf{y} = A\mathbf{x}$ for some vector $\mathbf{x}$. Show that the sum of the components of $\mathbf{y}$ equals the sum of the components of $\mathbf{x}$ — e.g. if the components of $\mathbf{x}$ are populations, then A conserves the total population. (Hint: recall the proof from class that the product of two Markov matrices is a Markov matrix.)

Problem 3: In class we learned that any positive Markov matrix A has a dominant eigenvalue, λ (=1), in the sense that it is simple eigenvalue (not a repeated root) and larger than the absolute value of any other eigenvalues. As a consequence, we know that any vector $\mathbf{x}$ will approach a multiple of the corresponding eigenvector $\mathbf{v}_1$ when we apply A to $\mathbf{x}$ again and again. In fact, this property holds for general matrix with positive entries, no matter whether is Markov or not.
(a) Use MATLAB to construct a random 5×5 positive matrix (`A = rand(5,5)` will give you a positive matrix), and use `[S,D]=eig(A)` to find its eigenvalues `diag(D)` and corresponding eigenvectors (columns of S). What is the dominant eigenvalue? Do the same procedure three times more, with 6×6, 7×7, and 8×8 random positive matrices respectively.
(b) You should notice that the dominant eigenvector has components that are all of the same sign (and hence they could all be chosen positive). (This generalizes the property, from class, that the dominant eigenvector of a Markov matrix has components that can be chosen nonnegative.) Prove that this is true in general. (Hint: the dominant eigenvector, when multiplied by A, should grow faster than any other vector. Show that if the dominant eigenvector had two components with different signs, that you could construct a different vector that grows faster when multiplied by A.)
(c) Is there any consistent pattern in the signs of the entries of all other eigenvectors?

Problem 4: The purpose of this problem is to re-examine some of the things we did earlier in the course, and to see what changes when we allow the possibility of complex vectors $\mathbf{x}$ and matrices A and the adjoint A^H. Justify your answers.
(a) For a complex A, is the left nullspace $N(A^T)$ orthogonal to $C(A)$ under the old unconjugated inner product $\mathbf{x}^T\mathbf{y}$ or the new conjugated inner product $\mathbf{x}^H\mathbf{y}$? What about $N(A^H)$ and $C(A)$?
(b) For a real vector subspace, V, the intersection of V and $V^\perp$ is only the single point $\mathbf{0}$. Now suppose V is a complex vector subspace. If we define $V^\perp$ as the set of vectors $\mathbf{x}$ with $\mathbf{x}^T\mathbf{v} = 0$ for all $\mathbf{v} \in V$, give an example of a V that intersects $V^\perp$ at a non-zero vector (hint: the simplest example is probably a 1-dimensional subspace of $\mathbb{C}^2$). What about if we use $\mathbf{x}^H\mathbf{v} = 0$, does V ever intersect $V^\perp$ at a nonzero vector using the conjugated definition of orthogonality? (Mathematicians use the latter definition of $V^\perp$.)
(c) Using your answer to (b), find an example of an $m \times n$ complex matrix A (for any m and n you like) such that $C(A)+N(A^T) \neq \mathbb{C}^m$, unlike for real matrices where $C(A) + N(A^T) = \mathbb{R}^m$ always (because the two subspaces only intersected at $\mathbf{0}$).
(d) Based on (a), (b), and (c), what would you suggest that we use as the four fundamental subspaces for complex matrices?
(e) We know A and A^T have equal rank. What about A and $\bar{A}$? A and A^H?
(f) How is $\det A$ related to $\det A^H$?

Problem 5: Justify the following true statements:
(a) If A is unitary, then A is invertible and A^{-1} is unitary.
(b) If A and B are unitary, then their product AB is unitary.
(c) If A is Hermitian and A is invertible, then A^{-1} is also Hermitian.
(d) If A is diagonalizable, then e^A is diagonalizable.

Problem 6: Suppose A is anti-Hermitian, i.e., $A^H = -A$. (This is also called "skew-Hermitian.") Note that a special case of this is a real anti-symmetric matrix, i.e. $A^T = -A$ for real A.)
(a) Show that iA is Hermitian. Conclude that the eigenvalues of A purely imaginary.
(b) Show that e^A is a unitary matrix.
(c) Show that the solution $\mathbf{u}(t)$ of the system $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ satisfies $\|\mathbf{u}(t)\|^2 = \|\mathbf{u}(0)\|^2$.

Problem 7: Unlike the exponential function on numbers, the matrix $e^A e^B$ is in general different from $e^B e^A$, and both can be different from e^{A+B}. Check the above statement for $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Problem 8: Solve the ODE system $\frac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{u}$ for $\mathbf{u}|_{t=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.