18.06 Problem Set 3

Due Wednesday, 26 September 2007 at 4 pm in 2-106.

Problem 1: A vector space is by definition a nonempty set V (whose elements are called vectors) together with rules of addition ($u, v \in V \Rightarrow u + v \in V$) and scalar multiplication ($k \in \mathbb{R}, v \in V \Rightarrow kv \in V$) which satisfy the eight conditions at the beginning of Problem Set 3.1 (P 118). Check whether the following sets with giving operations are vector spaces. (You should give reasons for your answer.)
(a) V is the set of all 2×2 symmetric matrices, with usual matrix addition and scalar multiplication.
(b) V is the set of all 2×2 invertible matrices, with usual matrix addition and scalar multiplication.
(c) V is the set $\{(x, y, z) \in \mathbb{R}^3 \mid x \leq y + z\}$, with usual vector addition and scalar multiplication.
(d) V is the set of polynomials whose degree are less or equal than 2, with usual function addition and scalar multiplication.
(e) $V = \{(a, b) \mid a, b \in \mathbb{R}\}$, with $(a, b) + (c, d) = (a + c, b + d)$ as usual, while $k(a, b) = (ka, 0)$.

Problem 2: Do problem 21 from section 3.2 (P 132) in your book.

Problem 3: (a) Do problem 12 from section 3.3 (P 142) in your book.
(b) Do problem 15 from section 3.3 (P 143) in your book.

Problem 4: Do problem 1 from section 3.4 (P 152) in your book.

Problem 5: Let $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{pmatrix}$, one can easily see that the column space of A is 2 dimensional, i.e. a plane. Recall that the equation of a plane through the origin is $\mathbf{b} \cdot \mathbf{n} = 0$, described by some normal vector $\mathbf{n}$—thus, if you knew $\mathbf{n}$, you would have a quick way to test whether any given $\mathbf{b}$ is in the column space.
(a) Show that $\mathbf{n}$ is in the null space of A^T. (Hint: $\mathbf{b} = A\mathbf{x}$ lies in the column space for arbitrary $\mathbf{x}$.)
(b) Find an $\mathbf{n}$ for this A.
(c) What is the analogous equation(s) for the column space of a 4×4 A with a 2d column space? (equations that you can use to quickly test whether any random $\mathbf{b}$ is in the column space)

Problem 6: (a) Describe the column space of $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

(b) For which vectors (b_1, b_2, b_3) does the system $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ has a solution?

Problem 7: Suppose A is a $m \times n$ matrix, and the system $A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has no solution, while $A\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ has exactly 1 solution. Denote by r the rank of A.
(a) What are possible values of (m, n, r)?
(b) What are all solutions to the system $A\mathbf{x} = \mathbf{0}$?
(c) Find a matrix A satisfying these conditions.

Problem 8: For the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, find 2×2 matrices B and C such that rank of AB is 1, while rank of AC is 0.

Problem 9: Find complete solutions to

(a) $x + 2y + 3z = 4.$ (b) $\left\{ \begin{array}{l} x + 2y + 3z = 4 \\ 2x + 4y + 8z = 10 \end{array} \right.$ (c) $\left\{ \begin{array}{l} x + 2y + 3z = 4 \\ 2x + 4y + 8z = 10 \\ -x - 2y + z = 0. \end{array} \right.$

Problem 10: Suppose A is an $m \times n$ matrix with $m < n$. A *right inverse* of A is a matrix B such that $AB = I$.
(a) What must the dimensions (the height and width) of B and of I be?
(b) One can find a right inverse B by using **MATLAB** operation $A \setminus I$. In **MATLAB**, input $A = [-5\ 3\ 1; 4\ 0\ 2]$ and $I = eye(2)$ to define the matrices, then input the command **A\I**. What out put do you get?
(c) Now try calculating B another way, with **rref([A I])**. (This is the reduced row echelon form, the result of Gauss-Jordan elimination.) What do you get? Use your result to state another, different, B with $AB = I$. Why is B not unique?
(d) Explain why A has no left inverse.