18.06 Problem Set 3 - Solutions

Due Wednesday, 26 September 2007 at 4 pm in 2-106.

Problem 1: (10=2+2+2+2+2) A vector space is by definition a nonempty set V (whose elements are called vectors) together with rules of addition ($u, v \in V \Rightarrow u + v \in V$) and scalar multiplication ($k \in \mathbb{R}, v \in V \Rightarrow kv \in V$) which satisfy the eight conditions at the beginning of Problem Set 3.1 (P 118). Check whether the following sets with giving operations are vector spaces. (You should give reasons for your answer.)

(a) V is the set of all 2×2 symmetric matrices, with usual matrix addition and scalar multiplication.

Solution Yes. It is a subspace of M on page 112.

Since it is a subset of M, we only need to show that for $A, B \in V$, $A + B$ still lies in V, and kA lies in V for all real number k; in other words, we need to show that if A, B are symmetric, then $A + B$ is symmetric, and kA is symmetric. They are both obvious.

(b) V is the set of all 2×2 invertible matrices, with usual matrix addition and scalar multiplication.

Solution No, it is not a vector space.

In fact, if we take any 2×2 invertible matrix A, let $B = -A$, then B is also invertible. However, $A + B = 0$ is not invertible.

(c) V is the set $\{(x, y, z) \in \mathbb{R}^3 \mid x \leq y + z\}$, with usual vector addition and scalar multiplication.

Solution No, it is not a vector space.

Counterexample: $\mathbf{v} = (1, 2, 1) \in V$, but $-\mathbf{v} = (-1, -2, -1) \notin V$.

(d) V is the set of polynomials whose degree are less than or equal to 2, with usual function addition and scalar multiplication.

Solution Yes. It is a subspace of F on page 112.

The reason is the same as in part (a): we only need to show that if f, g are polynomials of degree not more than 2, then $f + g$ and kf are polynomials of degree not more than 2; and they are obvious.

(e) $V = \{(a,b) \mid a,b \in \mathbb{R}\}$, with $(a,b)+(c,d) = (a+c, b+d)$ as usual, while $k(a,b) = (ka, 0)$.

Solution No, it is not a vector space.

It doesn't satisfy the condition (5) on page 118: $1(a,b) = (a,0) \neq (a,b)$!

Problem 2: (10) Do problem 21 from section 3.2 (P 132) in your book.

Solution The construction is not unique.

Let $B = \begin{pmatrix} 2 & 3 \\ 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the matrix with the given vectors as its row vectors. Then we want A such that $AB = 0$, which is equivalent to $B^T A^T = 0$. In other words, the columns of A^T (i.e. the rows of A) lie in the null space of $B^T = \begin{pmatrix} 2 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix}$.

Now we solve the equation $B^T\mathbf{x} = 0$:

$$\begin{pmatrix} 2 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1/2 & 0 \\ 0 & -2 & -3/2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1/4 & 1/2 \\ 0 & 1 & 3/4 & -1/2 \end{pmatrix}$$

so x_3 and x_4 are free variables, and the special solutions to $B^T\mathbf{x} = 0$ are given by $\mathbf{s}_1 = \begin{pmatrix} 1/4 \\ -3/4 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{s}_2 = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \\ 1 \end{pmatrix}$.

We conclude that A should be some $m \times 4$ matrix with each row vector

$$\begin{pmatrix} a & b & c & d \end{pmatrix} = c_1 \begin{pmatrix} 1/4 & -3/4 & 1 & 0 \end{pmatrix} + c_2 \begin{pmatrix} -1/2 & 1/2 & 0 & 1 \end{pmatrix}$$

for some arbitrary c_1 and c_2. Finally since the nullspace of A is two dimensional, we see that $rank(A)$ is 2. Some examples:

$$\begin{pmatrix} 1/2 & -3/2 & 2 & 0 \\ -1 & 1 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 & -4 & -2 \\ 1 & -3 & 4 & 0 \\ 1 & -1 & 0 & -2 \end{pmatrix}$$

Problem 3: (10=5+5) (a) Do problem 12 from section 3.3 (P 142) in your book.

Solution

$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$: there are two pivots, so rank(A)=2, we should remove 1 column to get an invertible submatrix. Obviously we should not remove the third column, otherwise we will get a singular matrix. Thus

$$A \rightsquigarrow \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}.$$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \rightsquigarrow A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$: there are only one pivot, so rank(A)=1, and we should remove 2 columns and 1 row to get a one by one invertible submatrix. In other words, S is just any nonzero entry in A. All possible choices are

$$A \rightsquigarrow \begin{pmatrix} 1 \end{pmatrix} \text{ or } \begin{pmatrix} 2 \end{pmatrix} \text{ or } \begin{pmatrix} 3 \end{pmatrix} \text{ or } \begin{pmatrix} 4 \end{pmatrix} \text{ or } \begin{pmatrix} 6 \end{pmatrix}.$$

$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: there are two pivots, so rank(A)=2. The only choice to get an invertible submatrix is

$$A \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) Do problem 15 from section 3.3 (P 143) in your book.

Solution

$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$. There is only one pivot, so rank(A)=1.

$AB = \begin{pmatrix} 6 & 4 & 16 \\ 12 & 8 & 32 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 6 & 4 & 16 \\ 0 & 0 & 0 \end{pmatrix}$. There is only one pivot, so rank(AB)=1.

$AM = \begin{pmatrix} 1+2c & 2b+4bc \\ 2+4c & 2b+4bc \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1+2c & b(1+2c) \\ 0 & 0 \end{pmatrix}$. There are two cases:

if $c = -1/2$, then there is no pivot, so rank(AM)=0;

if $c \neq -1/2$, then there is one pivot, so rank(AM)=1.

Problem 4: (10) Do problem 1 from section 3.4 (P 152) in your book.

Solution

★ Step 1: Reduce $[A\ \mathbf{b}]$ to $[U\ \mathbf{c}]$

$$\begin{pmatrix} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & -1 & -1 & -2 & b_3 - b_1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{pmatrix}$$

★ Step 2: Find the conditions on $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ have a solution.

From the equation, the condition is $b_2 + b_3 - 2b_1 = 0$.

★ Step 3: Describe the column space of A.

1st description: The column space is the plane containing all combinations of the pivot columns $(2, 2, 2)$ and $(4, 5, 3)$.

2nd Description: The column space contains all vectors with $b_2 + b_3 - 2b_1 = 0$.

★ Step 4: Describe the nullspace of A.

The special solutions have free variables $x_3 = 1, x_4 = 0$ and then $x_3 = 0, x_4 = 1$:

$$\mathbf{s_1} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{s_2} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

The null space $N(A)$ contains all $x_n = c_1\mathbf{s_1} + c_2\mathbf{s_2}$.

★ Step 5: Find a particular solution to $A\mathbf{x} = (4, 3, 5)$ and then the complete solution.

A particular solution can be found by setting free variables to be all 0. This gives

$$\mathbf{x_p} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

The complete solution is given by $\mathbf{x} = x_p + x_n$.

★ Step 6: Reduce $[U\ c]$ to $[R\ d]$.

$$\begin{pmatrix} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Problem 5: (10=4+4+2) Let $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{pmatrix}$, one can easily see that the column space of A is 2 dimensional, i.e. a plane. Recall that the equation of a plane through the origin is $\mathbf{b} \cdot \mathbf{n} = 0$, described by some normal vector $\mathbf{n}$—thus, if you knew $\mathbf{n}$, you would have a quick way to test whether any given $\mathbf{b}$ is in the column space.

(a) Show that $\mathbf{n}$ is in the nullspace of A^T. (Hint: $\mathbf{b} = A\mathbf{x}$ lies in the column space for arbitrary $\mathbf{x}$.)

Solution

Since $A\mathbf{x}$ lies in the column space, $A\mathbf{x} \cdot \mathbf{n} = 0$. In other words, $(A\mathbf{x})^T\mathbf{n} = 0$. So $\mathbf{x}^T A^T \mathbf{n} = 0$. Since this is true for all vectors $\mathbf{x}$, we conclude that $A^T\mathbf{n} = 0$, i.e. $\mathbf{n}$ lies in the nullspace of A^T.

(b) Find an $\mathbf{n}$ for this A.

Solution

We reduce $[A\ b]$ to $[U\ c]$:

$$\begin{pmatrix} 1 & 3 & 5 & b_1 \\ 2 & 4 & 6 & b_2 \\ 3 & 5 & 7 & b_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 5 & b_1 \\ 0 & -2 & -4 & b_2 - 2b_1 \\ 0 & -4 & -8 & b_3 - 3b_1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & b_1 \\ 0 & -2 & -4 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{pmatrix}$$

The column space is all vectors $\mathbf{b}$ with $b_1 - 2b_2 + b_3 = 0$, thus we can take $\mathbf{n} = (1, -2, 1)$. (Any nonzero multiple of this $\mathbf{n}$ is also a solution.)

(c) What is the analogous equation(s) for the column space of a 4×4 A with a 2d column space? (equations that you can use to quickly test whether any random $\mathbf{b}$ is in the column space)

Solution

If A is 4×4 and the column space is a 2d plane, then we need to use two equations to describe a 2d plane in $\mathbb{R}^4$ (each equation eliminates one degree of freedom, reducing the dimension by 1). Just as in (a), the column space must be orthogonal to the null space of A^T, and this leads us to conclude that there must be at two special solutions $\mathbf{v}_1$ and $\mathbf{v}_2$ in the null space of A^T, so that we get two equations $\mathbf{b} \cdot \mathbf{v}_1 = \mathbf{0}$, $\mathbf{b} \cdot \mathbf{v}_2 = \mathbf{0}$. [We will show this more generally and formally in lecture 10; somewhat handwavy explanations are acceptable here.]

Problem 6: (10=5+5) (a) Describe the column space of $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Solution By definition the column space contains all vectors of the form

$$a \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b + 2c \\ a + 2b + c \\ 0 \end{pmatrix},$$

in other words, it contains all vectors $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$.

(It would also be acceptable to note that $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a special solution in the null space of A^T, and hence the column space is all vectors orthogonal to $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, using the result from problem 5.)

(b) For which vectors (b_1, b_2, b_3) does the system $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ has a solution?

Solution We have

$$\begin{pmatrix} 1 & 2 & 3 & b_1 \\ 0 & 0 & 2 & b_2 \\ 0 & 0 & 1 & b_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & b_1 \\ 0 & 0 & 1 & b_2/2 \\ 0 & 0 & 0 & b_3 - b_2/2 \end{pmatrix},$$

thus the condition for the system to have a solution is $b_3 = b_2/2$.

(We may also use the methods in 6(a) above to find the column space.)

Problem 7: (10=2+4+4) Suppose A is a $m \times n$ matrix, and the system $A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has no solution, while $A\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ has exactly 1 solution. Denote by r the rank of A.

(a) What are possible values of (m, n, r)?

Solution Since $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ lies in the column space of A, we know $m = 3$ and $r > 0$. Since $A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has no solution, we see that A is not invertible, thus $r < 3$. Thus $r = 1$ or $r = 2$. Finally since for different choices of above, the system $A\mathbf{x} = \mathbf{b}$ has 0 or 1 solutions, we conclude that $r = n$. So the possible values of (m, n, r) are $(3, 1, 1)$ and $(3, 2, 2)$.

(b) What are all solutions to the system $A\mathbf{x} = \mathbf{0}$?

Solution since $r = n$, A has full column rank and there are no free variables, and hence the only vector in the null space is $\mathbf{x} = \mathbf{0}$.

(c) Find a matrix A satisfying these conditions.

Solution The matrix A satisfying these conditions is not unique.

For $n = 1$, then A is merely a vector. Since $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ lies in the column space, we see that $A = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix}$, where a is any nonzero number.

For $n = 2$, A is a matrix with full column rank and the column space contains the vector $(0, 1, 0)$. Some choices of the matrix are given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 0 & 5 \end{pmatrix}, \quad \begin{pmatrix} 9 & 0 \\ 1 & 5 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ -2 & 2 \end{pmatrix}, \text{ etc.}$$

Problem 8: (10) For the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, find 2×2 matrices B and C such that rank of AB is 1, while rank of AC is 0.

Solution The choices of B and C are not unique.

For B: Let $B = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, then $AB = \begin{pmatrix} a+2b & c+2d \\ 3a+6b & 3c+6d \end{pmatrix} \rightsquigarrow \begin{pmatrix} a+2b & c+2d \\ 0 & 0 \end{pmatrix}$. Thus for AB to have rank 1, we need $a \neq -2b$ or $c \neq -2d$.

For C: As we have seen above, for AC to have rank 0, we need $a = -2b$ and $c = -2d$, in other words, $C = \begin{pmatrix} -2b & -2d \\ -b & d \end{pmatrix}$.

(Lazy/Smart choices: $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.)

Problem 9: (10=2+4+4) Find complete solutions to

(a) $x + 2y + 3z = 4$. (b) $\begin{cases} x + 2y + 3z = 4 \\ 2x + 4y + 8z = 10 \end{cases}$ (c) $\begin{cases} x + 2y + 3z = 4 \\ 2x + 4y + 8z = 10 \\ -x - 2y + z = 0. \end{cases}$

Solution

(a) Obviously y, z are free variables, and a particular solution is given by setting them to be 0, i.e. $\mathbf{x}_p = (4, 0, 0)$. For the nullspace part x_n with $\mathbf{b} = 0$, set the free variables y, z to $1, 0$ and also $0, 1$, we have $\mathbf{s}_1 = (-2, 1, 0)$ and $\mathbf{s}_2 = (-3, 0, 1)$. Thus the general solution is $\mathbf{x} = \mathbf{x}_p + c_1\mathbf{s}_1 + c_2\mathbf{s}_2 = (4 - 2c_1 - 3c_2, c_1, c_2)$.

(b) We have

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

thus a particular solution is given by $(1, 0, 1)$, and the special solution is given by $(-2, 1, 0)$. So the general solutions are $\mathbf{x} = (1 - 2c, c, 1)$.

(c) We still forward elimination on $[A \ \mathbf{b}]$,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ -1 & -2 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 4 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so the particular solution is given by $\mathbf{x}_p = (1, 0, 0)$, and special solution is given by $(-2, 1, 0)$, as above. So the general solutions are $\mathbf{x} = (1 - 2c, c, 1)$. (Notice that the third equation is just 2 times the second equation minus 5 times the first equation!)

Problem 10: (10=2+3+3+2) Suppose A is an $m \times n$ matrix with $m < n$. A *right inverse* of A is a matrix B such that $AB = I$.

(a) What must the dimensions (the height and width) of B and of I be?

Solution The identity matrix I is square matrix which has the same height as A, thus I must be an $m \times m$ matrix. The height of B has to equal the width of A, and the width of B has to equal the width of B, thus B is $n \times m$ matrix.

(b) One can find a right inverse B by using **MATLAB** operation $A \setminus I$. In **MATLAB**, input $A = [-5\ 3\ 1; 4\ 0\ 2]$ and $I = eye(2)$ to define the matrices, then input the command **A\I**. What out put do you get?

Solution

```
>> A=[-5 3 1;4 0 2]; I=eye(2); A\I

ans =
   -0.1429    0.0714
         0         0
    0.2857    0.3571
```

(c) Now try calculating B another way, with **rref([A I])**. (This is the reduced row echelon form, the result of Gauss-Jordan elimination.) What do you get? Use your result to state another, different, B with $AB = I$. Why is B not unique?

Solution

```
>> A=[-5 3 1;4 0 2]; I=eye(2); rref([A I])

ans =
   1.0000         0    0.5000         0    0.2500
        0    1.0000    1.1667    0.3333    0.4167
```

We may rewrite the output above as

$$\begin{pmatrix} -5 & 3 & 1 & 1 & 0 \\ 4 & 0 & 2 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1.0000 & 0 & 0.5000 & 0 & 0.2500 \\ 0 & 1.0000 & 1.1667 & 0.3333 & 0.4167 \end{pmatrix}$$

In other words, we have the following elimination for $[A\ \mathbf{b}]$ with $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

respectively,

$$\begin{pmatrix} -5 & 3 & 1 & 1 \\ 4 & 0 & 2 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1.0000 & 0 & 0.5000 & 0 \\ 0 & 1.0000 & 1.1667 & 0.3333 \end{pmatrix}$$

and

$$\begin{pmatrix} -5 & 3 & 1 & 0 \\ 4 & 0 & 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1.0000 & 0 & 0.5000 & 0.2500 \\ 0 & 1.0000 & 1.1667 & 0.4167 \end{pmatrix}.$$

From the first one, we get a particular solution to $A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, given by $x_p = (0, 0.3333, 0)$. From the second one, we get a particular solution to $A\mathbf{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, namely $y_p = (0.2500, 0.4167, 0)$. Thus if we let

$$B = \begin{pmatrix} 0 & 0.2500 \\ 0.3333 & 0.4167 \\ 0 & 0 \end{pmatrix},$$

we have $AB = I_2$. (Notice that this B is adding zeros to the the end of last two columns).

Since the rank of A is 2, which equals the height of A but less than the width of A, we see that the equations $A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A\mathbf{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ both have infinitely many solutions. Thus the right inverse is not unique.

(d) Explain why A has no left inverse.

Solution If A has a left inverse, i.e. $CA = I_3$, then C is 3×2 matrix. Suppose B is any right inverse, then we have $C = CI_3 = C(AB) = (CA)B = I_2B = B$, which is impossible since B and C has different dimensions.

Another proof: as noted in part (c), A does not have full column rank, which means that there are nonzero null space solutions $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. Regardless of what we choose C to be, $CA\mathbf{x}$ must also equal $\mathbf{0}$, and thus CA cannot equal I: A has no left inverse.

A third proof: Since rank(A) is no more than two, we see that for any 3×2 matrix C, rank of CA is no more than two (cf. prob 17 in p. 143). We conclude that CA cannot be the identity matrix.