18.06 Problem Set 10

Due **Thursday, 29 November** 2007 at 4 pm in 2-106.

Problem 1: Take any matrix A of the form $A = B^H C B$, where B has full column rank and C is Hermitian and positive-definite.
(a) Show that A is Hermitian.
(b) Show that A is positive-definite by showing that $\mathbf{x} \cdot (A\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$ (hint: very similar to how we showed that $B^H B$ is positive-definite, in class).
(c) Show that $A = D^H D$ for some D with full column rank. (Hint: use $\sqrt{C}$ as defined in an earlier problem set.)

Problem 2: Consider Poisson's equation $d^2 f/dx^2 = g(x)$ [$g(x)$ is given and you want to find $f(x)$]. In lecture, we studied this for the case where f (and g) belongs to the space of real functions on $x \in [0, 1]$ with $f(0) = f(1) = 0$: we solved it by expanding f and g in Fourier sine series and then inverting each eigenvalue. Now, you should see what happens in the space of functions with zero slope at the boundaries [$f'(0) = f'(1) = 0$], where the eigenfunctions of d^2/dx^2 gave the Fourier cosine series.
(a) What is the null space of d^2/dx^2? (Note that you should only consider functions in the vector space, i.e. with zero slope at $x = 0$ and $x = 1$.)
(b) What is the column space of d^2/dx^2, in terms of the Fourier cosine series? That is, if $d^2 f/dx^2 = g(x)$, and you write out the cosine series of $g(x)$, what are the possible coefficients? (Hint: start with the cosine series of $f(x)$, and see what happens to it when you take the second derivative—what possible right-hand-sides can you get?)
(c) Suppose that $g(x)$ is the function $g(x) = 1$ for $x < 1/2$ and $g(x) = -1$ for $x \geq 1/2$. Find the cosine series of $g(x)$, using the cosine-series formulas:

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$a_n = 2 \int_0^1 g(x) \cos(n\pi x)\, dx.$$

[The $n = 0$ term has a 1/2 factor in the first formula so that the second formula works for all n. The reason for the difference is just a matter of normalization: $\|\cos(0\pi x)\|^2 = 1$, but $\|\cos(n\pi x)\|^2 = 1/2$ for $n > 0$.] Hint: you should find that $a_n = 0$ for even n.
(d) Verify that $g(x)$ from (c) is in the column space from (b). Using your answer from (c), find the cosine series for $f(x)$ to satisfy Poisson's equation. $f(x)$ should be the sum of a particular solution plus an arbitrary nullspace solution, using your answer to (a).

(e) In Matlab, plot the first four nonzero terms of your $g(x)$ cosine series, and then plot the first 8 nonzero terms—verify that the cosine series is converging to $g(x)$ (except right at the point of the discontinuity). For example, if you put the coefficients in the variables `a0`, `a1`, and so on (e.g. `a0 = 1.234/pi`), then you can plot the first four terms of the Fourier cosine series with the command:

```
fplot(@(x) a0/2 + a1*cos(pi*x) + a2*cos(2*pi*x) + a3*cos(3*pi*x), [0,1])
```

(f) As in (e), but plot the first 4 and 8 non-zero terms of your solution $f(x)$ from (d) [just pick some value for the nullspace part of the solution]. Which series converges faster, the one for f or the one for g?

Problem 3: (1) Follow the steps in problem 13 on page 350 to show that A^T is always similar to A.
(2) Is A^H always similar to A? Justify your conclusion.

Problem 4: Let $A = \mathbf{u}\mathbf{v}^T$ be any rank-1 matrix.
(1) What is the dimension of $N(A)$? What is $C(A)$?
(2) Find all eigenvalues of A, assuming that $\mathbf{u}$ and $\mathbf{v}$ both have n components so that A is square.
(3) What are the singular values of A? What is an SVD for A?
(4) Construct a rank-1 matrix A so that $A(1,0,1)^T = (2,1)^T$.

Problem 5: (1) Find the SVD for $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$.
(2) Find the pseudoinverse B of A.
(3) Compute AB, BA, ABA, and BAB.
(4) What are ABA and BAB for a general matrix A and its pseudoinverse B, given the definitions of the SVD and pseudoinverse?

Problem 6: Given the SVD $A = U\Sigma V^H$.
(1) What is the SVD of A^H and A^{-1} (assuming A is invertible)?
(2) If the QR decomposition of A is $A = QR$, what is the SVD for R?