18.06 Problem Set 7

SOLUTIONS

1. Section 6.1, Problem 2

Answers: Eigenvalues: $\lambda_{1,2} = -1, 5$; eigenvectors: $v_{1,2} = (-2, 1), (1, 1)$. The matrix $A + I$ has the same eigenvectors and eigenvalues increased by 1, i.e. 0 and 6.

2. Section 6.1, Problem 12

Answers: P has $\lambda = 1, 0, 1$ with eigenvectors $(1, 2, 0)$, $(2, -1, 0)$, $(0, 0, 1)$. The sum of the first and the last eigenvector is $(1, 2, 1)$; it is also an eigenvector with $\lambda = 1$. Since $P^{100} = P$, we get the same answers for P^{100}.

3. Section 6.1, Problem 22

Answers: $(A - \lambda I)$ has the same determinant as $A^T - \lambda I = (A - \lambda I)^T$. $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$: different eigenvectors.

4. Section 6.1, Problem 28

Answers: The rank of A is 1; eigenvalues of A are 0, 0, 0, 4. The rank of C is 2; eigenvalues of C are 0, 0, 2, 2.

5. Section 6.1, Problem 36

Solution: For a 3 by 3 permutation matrix P, we have $P^6 = I$, so $(\det P)^6 = \det(P^6) = 1$, and hence $\det P$ is either 1 or -1. The pivots of P are all equal to 1, as row reduction simply exchanges the rows to make the identity matrix I. The trace if P is either 0, 1, or 3 (it can't be 2 because once there are two 1's on the main diagonal of P, the third 1 must also be on the main diagonal). If v is an eigenvector of P with $Pv = \lambda v$, then depending on whether $P^2 = I$ or $P^3 = I$, we have either $P^2 v = v = \lambda^2 v$, i.e. $\lambda^2 = 1$, or else $P^3 v = \lambda^3 v$, i.e. $\lambda^3 = 1$. Hence the eigenvalues of P can equal 1, -1, or the two non-real cube roots of 1.

6. Section 6.2, Problem 2

Answer: If $A = S\Lambda S^{-1}$, then $A^3 = S\Lambda^3 S^{-1}$ and $A^{-1} = S\Lambda^{-1}S^{-1}$.

7. Section 6.2, Problem 5

Answers: (a) False: don't know λ's;

(b) True;

(c) True;

(d) False: need eigenvectors of S.

8. Section 6.2, Problem 15

Answers: (a) True;

(b) False;

(c) False (might have 2 or 3 independent eigenvectors).

Section 6.2, Problem 22

Answers: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $A^k = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.