18.06 Problem Set 4

SOLUTIONS

1. Section 3.5, Problem 42

Solution: If the 5 by 5 matrix $[A\ b]$ is invertible, b is not a combination of the columns of A. If $[A\ b]$ is singular, and the 4 columns of A are independent, b is a combination of those columns.

2. Section 3.6, Problem 17

Answers: Row space = yz-plane; column space = xy-plane; nullspace = x-axis; left nullspace = z-axis. For $I + A$: row space = column space = $\mathbb{R}^3$; nullspaces contain only the zero vector.

3. Section 3.6, Problem 23

Answers: Row space basis: $(3, 0, 3)$, $(1, 1, 2)$; column space basis $(1, 4, 2)$, $(2, 5, 7)$; rank of A is only 2, hence A is not invertible.

4. Section 3.6, Problem 25

Answers: (a) True (same rank).

(b) False: take $A = [1\ 0]$.

(c) False (A can be invertible and also not symmetric).

(d) True.

5. Section 4.1, Problem 22

Answers: $(1, 1, 1, 1)$ is a basis for $P^{\perp}$; $A = [1\ 1\ 1\ 1]$ has the plane P as its nullspace.

6. Section 4.1, Problem 26

Answers: $A = \begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$, $A^{\perp}A = 9I$ is *diagonal*:

$$(A^{\perp}A)_{ij} = (\text{column } i \text{ of } A) \cdot (\text{column } j \text{ of } A).$$

7. (a) Follows from the fact that the row space of a matrix is the space orthogonal to the nullspace.

(b) The nullspace of the reduced row echelon form of a matrix is the same as the nullspace of the matrix itself. Hence R_A and R_B, the reduced row echelon forms of A and B, have the same nullspace. Suppose R_A and R_B are different. Then R_A contains a non-zero row that R_B does not contain, or vice versa. Without loss of generality, assume the former. Consider the last (lowest) non-zero row r of R_A not contained in R_B. This row yields a condition of the form

$$x_p + a_{p+1}x_{p+1} + a_{p+2}x_{p+2} + \ldots = 0, \qquad (*)$$

where x_p is a pivot variable and $a_{p+i} \neq 0$ only if x_{p+i} is a free variable in both R_A and R_B. If x_p is a free variable in R_B, then the nullspace of R_B contains a vector $(x_1, x_2, \ldots)$ that does not satisfy $(*)$ because we can assign an arbitrary value to x_p leaving x_{p+1}, x_{p+2}, $\ldots$ untouched, and still be able to find x_1, $\ldots$, x_{p-1} such that $(x_1, x_2, \ldots)$ is in the nullspace of R_B. Hence in this case the nullspace of R_B contains a vector not in the nullspace of R_A — a contradiction. If x_p is a pivot variable in R_B, then R_B contains a row r' that yields a condition of the form

$$x_p + b_{p+1}x_{p+1} + b_{p+2}x_{p+2} + \ldots = 0. \qquad (**)$$

Since rows r of R_A and r' of R_B are different, we have $a_{p+i} \neq b_{p+i}$ for some i, so by choosing an appropriate value of the free variable x_{p+i} we can find x_p, x_{p+1}, $\ldots$ satisfying one of $(*)$ or $(**)$ but not the other. Hence in this case the nullspaces of R_A and R_B are different — a contradiction.

8. (a)

$$A_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

$$A_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) $A_5^{-1} = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 & 0 \\ 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$;

$$A_6^{-1} = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}.$$

(c) If $A_5 = L_5 U_5$ and $A_6 = L_6 U_6$, then

$$L_5^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}; \quad L_6^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$