18.06 Professor Strang Final Exam December 19, 2005

Your PRINTED name is: ____________________

Please circle your recitation:

1)	M 2	2-131	P. Lee	2-087	2-1193	`lee`
2)	M 2	2-132	T. Lawson	4-182	8-6895	`tlawson`
4)	T 10	2-132	P-O. Persson	2-363A	3-4989	`persson`
5)	T 11	2-131	P-O. Persson	2-363A	3-4989	`persson`
6)	T 11	2-132	P. Pylyavskyy	2-333	3-7826	`pasha`
7)	T 12	2-132	T. Lawson	4-182	8-6895	`tlawson`
8)	T 12	2-131	P. Pylyavskyy	2-333	3-7826	`pasha`
9)	T 1	2-132	A. Chan	2-588	3-4110	`alicec`
10)	T 1	2-131	D. Chebikin	2-333	3-7826	`chebikin`
11)	T 2	2-132	A. Chan	2-588	3-4110	`alicec`
12)	T 3	2-132	T. Lawson	4-182	8-6895	`tlawson`

Grading

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1 (12 pts.) This question is about the matrix $A = I + E$ where E is the all-ones matrix $\texttt{ones}(4,4)$:

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} = I + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

(a) By elimination find the pivots of A.

(b) Factor A into LDL^{T} (if that is possible).

(c) The inverse matrix has the form $A^{-1} = I + cE$. Figure out E^2 and then choose the number c so that $AA^{-1} = I$.

2 (12 pts.) Keep the same matrix A as in Problem 1.

(a) Find the matrix P that projects any vector in $\mathbf{R}^4$ onto the subspace spanned by the first column of A.

(b) Describe the nullspace of $I - P$ and the nullspace of PA.

(c) Find all the eigenvalues of P.

3 (12 pts.) Now suppose $A = I + bE$, with the same $E = \texttt{ones}(4, 4)$.

(a) What are the eigenvalues of E?

(b) If $b = 2$, what is the determinant of A?

(c) Suppose you know that $x^{\mathrm{T}}Ax > 0$ for every nonzero vector x. (Same matrix A.) What are the possible values of b?

4 (16 pts.) Suppose A is an 8 by 8 invertible matrix. Throw away any 3 columns of A to get an 8 by 5 matrix B.

(a) You will correctly think that B has rank 5. Give a mathematical reason why this is true.

(b) Tell all you know about the nullspace of B^{T} and the reduced row echelon form `rref`(B).

(c) Give as much information as possible about the eigenvalues and eigenvectors of $B^{\mathrm{T}}B$ and BB^{T} (those are separate questions).

5 (12 pts.) Suppose Q is an m by n matrix with $Q^{\mathrm{T}}Q = I$. Write down the most important facts about

(a) The columns of Q

(b) m and n and the rank of Q

(c) The least squares solution $\widehat{x}$ to $Qx = b$

6 (12 pts.) (a) The eigenvalues of $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ are ______________________ .

(b) An orthogonal set of 4 eigenvectors is ______________________ .

(c) CIRCLE every class of matrices to which this matrix A belongs:

diagonalizable	permutation	nonsingular
Jordan matrix	orthogonal	projection
skew-symmetric		

7 (12 pts.) Suppose A is 2 by 3 with this Singular Value Decomposition $U\Sigma V^{\mathrm{T}}$. U and V are orthogonal matrices:

$$A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{\mathrm{T}} \\ v_2^{\mathrm{T}} \\ v_3^{\mathrm{T}} \end{bmatrix}.$$

(a) Find a basis for the nullspace of A.

(b) Find all solutions to the equation $Ax = u_1$.

(c) Find the shortest solution to $Ax = u_1$ (minimum length vector) and prove that it is shortest.

8 (12 pts.) Suppose A (3 by 3) has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and independent eigenvectors x_1, x_2, x_3.

(a) What is the general form of the solutions to $u_{k+1} = Au_k$ and $\dfrac{du}{dt} = Au$? (Two questions)

(b) Suppose every solution to $u_{k+1} = Au_k$ approaches a multiple $c\,x_1$ as $k \to \infty$ (c depends on u_0). What does this tell you about $\lambda_1, \lambda_2, \lambda_3$?

(c) For some 3 by 3 matrices, the complete solution to $\dfrac{du}{dt} = Au$ does *not* have the form you gave in part (a). What can go wrong? Give an example of such a matrix A.