18.06 Professor Strang Quiz 2 November 2, 2005

Grading

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Your PRINTED name is: ____________________

Please circle your recitation:

1)	M 2	2-131	P. Lee	2-087	2-1193	lee
2)	M 2	2-132	T. Lawson	4-182	8-6895	tlawson
4)	T 10	2-132	P-O. Persson	2-363A	3-4989	persson
5)	T 11	2-131	P-O. Persson	2-363A	3-4989	persson
6)	T 11	2-132	P. Pylyavskyy	2-333	3-7826	pasha
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8)	T 12	2-131	P. Pylyavskyy	2-333	3-7826	pasha
9)	T 1	2-132	A. Chan	2-588	3-4110	alicec
10)	T 1	2-131	D. Chebikin	2-333	3-7826	chebikin
11)	T 2	2-132	A. Chan	2-588	3-4110	alicec
12)	T 3	2-132	T. Lawson	4-182	8-6895	tlawson

1 (30 pts.) The matrix A has a varying $1 - x$ in the $(1,2)$ position:

$$A = \begin{bmatrix} 2 & \mathbf{1 - x} & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix}$$

(a) When $x = 1$ compute $\det A$. What is the $(1,1)$ entry in the inverse when $x = 1$?

Solution: We expand along the first row, then use elimination, and then expand along the first column:

$$\det \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix} = 2 \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = 2 \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{bmatrix} =$$

$$= 2 \cdot \det \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix} = 2 \cdot 2 = 4.$$

We use the cofactor formula to find $A^{-1}(1,1)$:

$$A^{-1}(1,1) = \frac{(-1)^{1+1}}{\det A} \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \frac{2}{4} = \frac{1}{2}.$$

(b) When $x = 0$ compute $\det A$.

Solution: Subtract the second column from the first, and then expand along the first column:

$$\det \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 3 & 9 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = 2.$$

(c) How do the properties of the determinant say that $\det A$ is a linear function of x? For any x compute $\det A$. For which x's is the matrix singular?

Solution: To find $\det A(x)$, we perform the same steps as in the previous part:

$$\det \begin{bmatrix} 2 & 1-x & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix} = \det \begin{bmatrix} 1+x & 1-x & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 3 & 9 \end{bmatrix} =$$

$$= (1+x) \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = 2 + 2x.$$

The matrix $A(x)$ is singular when $\det A(x) = 0$, i.e. when $x = -1$.

2 (30 pts.) This matrix Q has orthonormal columns q_1, q_2, q_3:

$$Q = \begin{bmatrix} .1 & .5 & a \\ .7 & .5 & b \\ .1 & -.5 & c \\ .7 & -.5 & d \end{bmatrix}.$$

(a) What equations must be satisfied by the numbers a, b, c, d? Is there a unique choice for those numbers, apart from multiplying them all by -1?

Solution: We need the vector (a, b, c, d) to be a unit vector orthogonal to the other two columns of Q. Hence the numbers a, b, c, and d must satisfy the following equations:

$$a^2 + b^2 + c^2 + d^2 = 1;$$

$$.1a + .7b + .1c + .7d = 0;$$

$$.5a + .5b - .5c - .5d = 0.$$

The last two equations define a two-dimensional subspace of $\mathbb{R}^4$. Taking two independent vectors in this subspace and normalizing them gives two good choices of a, b, c, and d that are not merely negatives of each other. Hence the choice of the last column of Q is not unique.

(b) Why is $P = QQ^{\mathrm{T}}$ a projection matrix? (Check the two properties of projections.) Why is QQ^{T} a singular matrix? Find the determinants of $Q^{\mathrm{T}}Q$ and QQ^{T}.

Solution: P is a projection matrix if it satisfies two conditions: P is symmetric, and $P^2 = P$. And indeed, we have

$$P^{\mathrm{T}} = (QQ^{\mathrm{T}})^{\mathrm{T}} = (Q^{\mathrm{T}})^{\mathrm{T}}Q^{\mathrm{T}} = QQ^{\mathrm{T}} = P;$$

$$P^2 = Q(Q^{\mathrm{T}}Q)Q^{\mathrm{T}} = QQ^{\mathrm{T}} = P$$

since $Q^{\mathrm{T}}Q = I$ by orthonormality of Q. (Another valid argument is that P is the matrix for projection onto the column space of Q, as can be easily checked using the projection matrix formula.)

The matrix $P = QQ^{\mathrm{T}}$ is singular because both Q and Q^{T} have rank 3, and hence their product P has rank at most 3 (in fact, exactly 3). Since P is a 4 by 4 matrix, it is not full rank and hence singular.

Since QQ^{T} is singular, we have $\det QQ^{\mathrm{T}} = 0$. Also, $\det Q^{\mathrm{T}}Q = \det I = 1$.

(c) Suppose Gram-Schmidt starts with those same first two columns and with the third column $a_3 = (1, 1, 1, 1)$. What third column would it choose for q_3? You may leave a square root not completed (if you want to).

Solution: The Gram-Schmidt process first produces a vector Q_3 orthogonal to both q_1 and q_2:

$$Q_3 = a_3 - (q_1 \cdot a_3)q_1 - (q_2 \cdot a_3)q_2$$

(remember that q_1 and q_2 are unit vectors). We have $q_2 \cdot a_3 = 0$ and $q_1 \cdot a_3 = 1.6$, so

$$Q_3 = (1, 1, 1, 1) - 1.6 \cdot (.1, .7, .1, .7) = (.84, -.12, .84, -.12).$$

Normalizing, we get

$$q_3 = \frac{Q_3}{||Q_3||} = (.7, -.1, .7, -.1).$$

3 (40 pts.) Our measurements at times $t = 1, 2, 3$ are $b = 1, 4$, and b_3. We want to fit those points by the nearest line $C + Dt$, using least squares.

(a) Which value for b_3 will put the three measurements on a straight line? *Which line is it*? Will least squares choose that line if the third measurement is $b_3 = 9$? (Yes or no).

Solution: The three data points lie on the same line when $b_3 = 7$. This line is $-2 + 3t$. If $b_3 = 9$, the least squares method will NOT choose this line. (A quick way to see this is from the fact that the line chosen by least squares will give the average of the given b's at the time equal to the average of the given t's; in this case, the best fit line would take the value $(1 + 3 + 9)/3 = 13/3$ at $t = (1 + 2 + 3)/3 = 2$, whereas our line gives 4 at $t = 2$.)

(b) What is the linear system $Ax = b$ that would be solved exactly for $x = (C, D)$ if the three points do lie on a line? Compute the projection matrix P onto the column space of A. Remember the inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Solution: The linear system for $x = (C, D)$ would be the following:

$$Ax = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} x = \begin{bmatrix} 1 \\ 4 \\ b_3 \end{bmatrix}.$$

We compute the projection matrix P onto the column space of A using the projection matrix formula:

$$P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = \frac{1}{6} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} =$$

$$\frac{1}{6}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 8 & 2 & -4 \\ -3 & 0 & 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}.$$

(c) What is the rank of that projection matrix P? How is the column space of P related to the column space of A? (You can answer with or without the entries of P computed in (b).)

Solution: The column space of P is the space consisting of all the vectors Pb, i.e. all the projections of vectors in $\mathbb{R}^3$ onto the column space of A, which is precisely the column space of A. Thus the rank of P is equal to the rank of A, which is 2.

(d) Suppose $b_3 = 1$. Write down the equation for the best least squares solution $\widehat{x}$, and show that the best straight line is horizontal.

Solution: The equation for the best least squares solution $\widehat{x}$ is $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$, where $b = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$. Writing out this system, we get

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}\widehat{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}.$$

The solution to this system is $\widehat{x} = \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, so the best fit line is the horizontal line $b = 2$.