18.06, Fall 2004, Problem Set 8 Solutions

1. (7 pts.) Let

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ 4 & 1 & 0 \end{bmatrix}.$$

(a) C_{ij} is given by $(-1)^{i+j}\det(M_{ij})$ where M_{ij} is obtained from A by removing row i and column j.

$$C = \begin{bmatrix} -1 & 4 & -1 \\ 2 & -8 & 4 \\ 1 & -2 & 1 \end{bmatrix}.$$

(b) Using the cofactor formula, say along row 1, we get:

$$\det(A) = 0 \cdot (-1) + 1 \cdot 4 + 2 \cdot (-1) = 2.$$

(c) We have that

$$A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{2} \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -2 \\ -1 & 4 & 1 \end{bmatrix}.$$

We verify that

$$AA^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -2 \\ -1 & 4 & 1 \end{bmatrix} = I.$$

(d)

$$\det(-3A^4) = (-3)^3 \det(A^4) = -27 \det(A)^4 = -27 \cdot 16 = -432.$$

2. (5 pts.) The system is $Ax = b$ where

$$A = \begin{bmatrix} 2 & 1 & -1 & 1 \\ -1 & 2 & 1 & 0 \\ 0 & -3 & 1 & -1 \\ 1 & 7 & 0 & 3 \end{bmatrix},$$

and

$$b = \begin{bmatrix} 5 \\ -6 \\ 1 \\ -1 \end{bmatrix}.$$

Cramer's rule says that

$$x_3 = \frac{\begin{vmatrix} 2 & 1 & 5 & 1 \\ -1 & 2 & -6 & 0 \\ 0 & -3 & 1 & -1 \\ 1 & 7 & -1 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & -1 & 1 \\ -1 & 2 & 1 & 0 \\ 0 & -3 & 1 & -1 \\ 1 & 7 & 0 & 3 \end{vmatrix}} = \frac{-8}{8} = -1.$$

3. (5 pts.) The fifth pivot is given by:

$$\frac{\begin{vmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 2 & 4 & 6 & 0 \\ -1 & -2 & 2 & 4 & 0 \\ 5 & 4 & 3 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 6 \\ -1 & -2 & 2 & 4 \\ 5 & 4 & 3 & 1 \end{vmatrix}}.$$

Using the cofactor formula for the determinant in the numerator along the last column, we see that the numerator is twice the denominator, thus the fifth pivot is equal to 2.

4. (4 pts.) The two rows must be multiples of each other. Thus the first row must equal the second one times x. Hence we get that $y = x^2$ and $8 = xy$. From this we derive that $8 = x^3$ and $y = x^2$ which has 3 solutions: $(x, y) = (2, 4)$, $(x, y) = (2e^{i2\pi/3}, 4e^{i2\pi/3})$ and $(x, y) = (2e^{i4\pi/3}, 4e^{i4\pi/3})$.

5. (6 pts.)

The characteristic polynomial for A is:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 1,$$

and its roots (thus the eigenvalues) are: $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$.

The eigenvector corresponding to $\lambda_1 = 2+i$ is a non-zero vector of the nullspace of $A-(2+i)I$:

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}.$$

We can take $v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ or any non-zero (even complex) scalar multiple of it.

The eigenvector corresponding to $\lambda_2 = 2-i$ is a non-zero vector of the nullspace of $A-(2-i)I$:

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}.$$

We can take $v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ or any non-zero (even complex) scalar multiple of it.

6. (7 pts.) Let

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}.$$

(a)

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} \\ &= (2-\lambda)[(1-\lambda)(-1-\lambda) - 3] + 2(-1-\lambda-1) + 3(3-1+\lambda) \\ &= (2-\lambda)(\lambda^2 - 4) + 2 + \lambda \\ &= -\lambda^3 + 2\lambda^2 + 5\lambda - 6 \end{aligned}$$

(b) Substituting $\lambda = 1$ in the characteristic polynomial, we observe that it is indeed a root of it, and therefore an eigenvalue. For an eigenvector we need to look at the nullspace of $A - I$:

$$A - I = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{bmatrix}.$$

Doing row eliminations, we first get:

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & 5 & -5 \end{bmatrix},$$

and then

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The special solution, and thus an eigenvector corresponding to $\lambda = 1$, is then:

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

(c) To find the other eigenvalues we can first factor $\lambda - 1$ from the characteristic polynomial, and then find the remaining roots:

$$-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = -(\lambda - 1)(\lambda^2 - \lambda - 6).$$

The roots of the quadratic $\lambda^2 - \lambda - 6$ are 3 and -2, and these are the remaining eigenvalues.

7. (6 pts.)

   (a) The easiest way (or the least difficult...) to argue this is the following. Let's assume that λ is an eigenvalue of a permutation matrix P and v is the corresponding eigenvector. Thus $Pv = \lambda v$. Both v and λ can be complex (and they typically are). Now look at the sum of the modulus of the v_i's, i.e. $\sum_i |v_i|$. As P is a permutation matrix, Pv has the same entries as v except that they are in a different order. Thus

$$\sum_i |(Pv)_i| = \sum_i |v_i|. \tag{1}$$

   But, for two complex numbers a and b, we have that $|ab| = |a| \cdot |b|$. Thus $\sum_i |(Pv)_i| = \sum_i |\lambda v_i| = |\lambda| \sum_i |v_i|$. Combining this with (1), we get that $|\lambda| = 1$.

   (b) Let

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

   Let $\lambda_1, \lambda_2, \cdots, \lambda_5$ be its eigenvalues. There are several ways again to compute the eigenvalues. One could find the characteristic polynomial and find its roots. Here is another shorter way.

   We know that all eigenvalues have modulus 1, so they can be expressed as $e^{i\theta_i}$. Observe that

$$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

   The eigenvalues of P^2 must be $\lambda_1^2, \lambda_2^2, \cdots, \lambda_5^2$. But 1 is clearly an eigenvalue of P^2 with multiplicity 2 (both geometric and algebraic) as both $v_1 = (1, 0, 0, 0, 0)$ and $v_2 = (0, 1, 0, 0, 0)$ are vectors[1] such that $P^2 v_1 = v_1$ and $P^2 v_2 = v_2$. This implies that $\lambda_1^2 = \lambda_2^2 = 1$ and thus $\lambda_1 \in \{-1, 1\}$ and $\lambda_2 \in \{-1, 1\}$.

   Now P^3 is:

$$P^3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

[1] There is a subtlety here. v_1 and v_2 are eigenvectors of P^2 but are not eigenvectors of P. In lecture, we had seen that P and P^2 would share eigenvectors. Well, they do. Indeed for P^2 there are many choices for the linearly independent eigenvectors as the nullspace for $P^2 - I$ has dimension 2. And one *can* find (complex) eigenvetors of P^2 corresponding to $\lambda = 1$ such that they are also complex eigenvectors corresponding to these 2 eigenvalues of P.

But $v_3 = (0,0,1,0,0)$, $v_4 = (0,0,0,1,0)$ and $v_5 = (0,0,0,0,1)$ are eigenvectors of P^3 and they are linearly independent on v_1 and v_2 and so they correspond to 3 more eigenvalues. Thus we have that $\lambda_3^3 = 1 = \lambda_4^3 = \lambda_5^3$. Thus possible values for λ_3, λ_4 and λ_5 are $1, e^{i2\pi/3}, e^{i4\pi/3}$.

But we know also that the sum of the eigenvalues of P is equal to the trace of P, which is 0. Thus the only possibility for all the λ_i's is now: $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 1$, $\lambda_4 = e^{i2\pi/3}$ and $\lambda_5 = e^{i4\pi/3}$.