18.06, Fall 2004, Problem Set 7

Due before 4PM on Wednesday November 3rd, 2004, in the boxes in 2-106. No late homework will be accepted. Don't forget to write your name, recitation section **and the names of students you have collaborated with** on the problem set. There is one box for each recitation section. For full credit, please be sure to show and explain your work. Exercises refer to the 3rd edition of the textbook.

Reading assignment: Sections 7.3, Chapter 5.

Given the Red Sox World Series win, you can skip one of the problems below and get full credit on it. :-) Just write Red Sox as your answer (so that we know which one you get for free.) If you are from St. Louis, you are welcome to solve all of them.

1. Let
$$\mathbf{a_1} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \mathbf{a_2} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \mathbf{a_3} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
and
$$\mathbf{b_1} = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix}, \mathbf{b_3} = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}.$$

   (a) Show that $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ and $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$ are bases of the *same* subspace F of the vector space of all 2×2 matrices.

   (b) Derive the matrix L which allows to go from a representation in the basis $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ to a representation in the basis $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{b_3}$. (This means that if $\mathbf{v} = c_1\mathbf{a_1} + c_2\mathbf{a_2} + c_3\mathbf{a_3} = d_1\mathbf{b_1} + d_2\mathbf{b_2} + d_3\mathbf{b_3}$ then $d = Lc$.)

   (c) Verify your matrix L by finding the representations of
$$\mathbf{v} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$
   in both bases and checking that indeed $d = Lc$.

2. For each class of matrices below, list all possible values for the determinant with an example of a matrix in the class achieving each value. Also argue that no other values are possible than the ones you list.

   (a) the class of $n \times n$ permutation matrices P,

   (b) the class of orthogonal matrices Q (i.e. square matrices Q with $Q^TQ = I$) ,

   (c) the class of projection matrices,

   (d) the class of 2×2 rotation matrices $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

3. Use elementary (row and/or column) operations to show that the value of the following 4×4 determinant
$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$$

is
$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

4. Find a value for x such that
$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ x & 2 & 3 & 4 \\ 7 & 0 & 5 & 6 \\ 8 & 0 & 0 & 3 \end{vmatrix} = 10.$$
(This does not require involved calculations.)

5. Show that there does not exist a 7×7 matrix A which is both non-singular and with the property that $A^T = -A$. (A 2×2 such matrix would be for example $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.)