18.06, Fall 2004, Problem Set 5 Solutions

1. (10 pts.)

The (edge-node) incidence matrix of the graph is (in MATLAB output) the following 12×8 matrix:

```
A =

    -1     1     0     0     0     0     0     0
    -1     0     1     0     0     0     0     0
     0     0    -1     1     0     0     0     0
     0    -1     0     1     0     0     0     0
    -1     0     0     0     0     0     1     0
     0     0    -1     0     1     0     0     0
     0     0     0    -1     0     1     0     0
     0    -1     0     0     0     0     0     1
     0     0     0     0     1     0    -1     0
     0     0     0     0    -1     1     0     0
     0     0     0     0     0    -1     0     1
     0     0     0     0     0     0    -1     1
```

The diagonal matrix C of conductances is

```
>> C=diag([1 1 2 1 1 2 2 1 1 2 1 1])

C =

     1     0     0     0     0     0     0     0     0     0     0     0
     0     1     0     0     0     0     0     0     0     0     0     0
     0     0     2     0     0     0     0     0     0     0     0     0
     0     0     0     1     0     0     0     0     0     0     0     0
     0     0     0     0     1     0     0     0     0     0     0     0
     0     0     0     0     0     2     0     0     0     0     0     0
     0     0     0     0     0     0     2     0     0     0     0     0
     0     0     0     0     0     0     0     1     0     0     0     0
     0     0     0     0     0     0     0     0     1     0     0     0
     0     0     0     0     0     0     0     0     0     2     0     0
     0     0     0     0     0     0     0     0     0     0     1     0
     0     0     0     0     0     0     0     0     0     0     0     1
```

If we denote the potentials by x, the currents by y and the current inputs by $f = (-1, 0, 0, 0, 0, 0, 0, 1)$, we first need to solve $A^T C A x = f$ to get x and then compute $y = CAx$. (We have used the notation and the equations developed in lecture and in the book, although, strictly speaking, current is proportional to the potential *drop* and not potential increase. Answers with all the currents of the opposite sign are acceptable too.) We saw in lecture and in the book that

A^TCA is a $n \times n$ matrix of rank $n-1$ ($n = 8$ here), and that its nullspace is the line spanned by the vector of all 1's. Thus $A^TCAx = f$ determines only x up to an additive constant. We can therefore fix any x_i to be any value, say fix $x_8 = 0$. Letting $B = A^TCA$, this means that we can remove the last column of B (since $x_8 = 0$) and also any row of B (since the system has rank 7), say the last row of B. We can therefore get x as follows:

```
>> B=A'*C*A

B =

     3    -1    -1     0     0     0    -1     0
    -1     3     0    -1     0     0     0    -1
    -1     0     5    -2    -2     0     0     0
     0    -1    -2     5     0    -2     0     0
     0     0    -2     0     5    -2    -1     0
     0     0     0    -2    -2     5     0    -1
    -1     0     0     0    -1     0     3    -1
     0    -1     0     0     0    -1    -1     3

>> G=B(1:7,1:7)

G =

     3    -1    -1     0     0     0    -1
    -1     3     0    -1     0     0     0
    -1     0     5    -2    -2     0     0
     0    -1    -2     5     0    -2     0
     0     0    -2     0     5    -2    -1
     0     0     0    -2    -2     5     0
    -1     0     0     0    -1     0     3

>> inv(G)

ans =

    0.7143    0.3571    0.4286    0.3571    0.3571    0.2857    0.3571
    0.3571    0.5617    0.3052    0.3279    0.2565    0.2338    0.2045
    0.4286    0.3052    0.6753    0.4870    0.4870    0.3896    0.3052
    0.3571    0.3279    0.4870    0.6266    0.4123    0.4156    0.2565
    0.3571    0.2565    0.4870    0.4123    0.6266    0.4156    0.3279
    0.2857    0.2338    0.3896    0.4156    0.4156    0.5325    0.2338
    0.3571    0.2045    0.3052    0.2565    0.3279    0.2338    0.5617

>> x = inv(G)*[-1; 0; 0; 0; 0; 0; 0]
```

```
x =

    -0.7143
    -0.3571
    -0.4286
    -0.3571
    -0.3571
    -0.2857
    -0.3571
```

We can now get the currents y from $y = CAx$ (and remembering that x_8 is 0):

```
y=C*A(:,1:7)*x

y =

   0.3571
   0.2857
   0.1429
   0.0
   0.3571
   0.1429
   0.1429
   0.3571
   0.0
   0.1429
   0.2857
   0.3571
```

2. (6 pts)

   In exercise 3 of problem set 3, we argued that $C(AB) \subseteq C(A)$. This implies that $rank(AB) \leq rank(A)$ as the dimension of the column space of a matrix is equal to its rank.

   Taking transposes, the statement $C(PQ) \subseteq C(P)$ also implies that $R(Q^T P^T) \subseteq R(P^T)$. Letting $A = Q^T$ and $B = P^T$, we get $rank(AB) \leq rank(B)$.

3. (16 pts.)

   (a) We can simply take the n unit vectors along the coordinate axes; these are n vectors with all components equal to 0, except one component equal to 1.

   (b) i. Entry (i, j) of the matrix R is the dot product of q_i and q_j, where q_i denotes the ith column of Q. As all the pair-wise distances are 1, we have that, for $i \neq j$, $1 = ||q_i - q_j||^2 = ||q_i||^2 + ||q_j||^2 - 2q_i \cdot q_j = 2 - 2q_i \cdot q_j$, implying that $q_i \cdot q_j = \frac{1}{2}$. Thus, we must have

$$R_{ij} = \begin{cases} 1 & \text{for } i = j \\ 1/2 & \text{for } i \neq j \end{cases}$$

For $p = 5$ for exeample, R would be the following 4×4 matrix:

$$R = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

ii. R has full rank, i.e. $rank(R) = p - 1$. This can be shown in many ways. One could "guess" that the inverse of R is of the form:

$$R^{-1}_{ij} = \begin{cases} a & \text{for } i = j \\ b & \text{for } i \neq j \end{cases}$$

Then, for this matrix to be the inverse of R, we would need to be able to find a and b such that (just using the fact that $RR^{-1} = I$):

$$\begin{cases} a + \frac{p-2}{2} b = 1 \\ \frac{a}{2} + \frac{p-1}{2} b = 0 \end{cases}$$

which has $a = \frac{2(p-1)}{p}$ and $b = -\frac{2}{p}$ as solution. Thus R has an inverse and is of full rank. For example, for $p = 5$, R^{-1} is given by:

$$R^{-1} = \begin{bmatrix} 1.6 & -0.4 & -0.4 & -0.4 \\ -0.4 & 1.6 & -0.4 & -0.4 \\ -0.4 & -0.4 & 1.6 & -0.4 \\ -0.4 & -0.4 & -0.4 & 1.6 \end{bmatrix}.$$

Another way to show that R has full rank is to show that the only vector in $N(R)$ is the 0 vector. Indeed if $Rx = 0$ we have that $\frac{x_j}{2} + \frac{1}{2}\sum_{i=1}^{p-1} x_i = 0$ for all $j = 1, \cdots, p-1$, which implies that $x_1 = x_2 = \cdots = x_{p-1}$ which in turn implies that all the x_i's are 0. Thus $dim(N(R)) = 0$ and R has full rank.

iii. Since $R = Q^T Q$ we know by the previous exercise that $rank(R) \leq rank(Q)$. But $rank(R) = p - 1$ (by the previous subquestion) and $rank(Q) = n$ as the columns of Q span R^n. Thus we have $p - 1 \leq n$ or $p \leq n + 1$.

4. (8 pts.)

Here is some possible Matlab code to do the exercise:

```
for I=1:50,
A=randn(50,30);
P=A*inv(A'*A)*A';
W=P;
rank(W);
Q=W'*W;
d=diag(Q);
e=sqrt(d*ones(1,50)+ones(50,1)*d'-2*Q);
ma=max(max(e));
```

```
mi=min(min(e+ma*eye(50)));
C(I)=ma/mi;
end
>> min(C)

ans =

    1.5903
```

By repeating many times, one could get below 1.5 but any answer below 1.75 is valid.